I.3 THE PHYSICAL CONTENT OF QUANTUM KINEMATICS AND MECHANICS

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First we define the terms *velocity*, *energy*, etc. (for example, for an electron) which remain valid in quantum mechanics. It is shown that canonically conjugate quantities can be determined simultaneously only with a characteristic indeterminacy (§1). This indeterminacy is the real basis for the occurrence of statistical relations in quantum mechanics. Its mathematical formulation is given by the Dirac-Jordan theory (§2). Starting from the basic principles thus obtained, we show how microscopic processes can be understood by way of quantum mechanics (§3). To illustrate the theory, a few special *gedankenexperiments* are discussed (§4).

We believe we understand the physical content of a theory when we can see its qualitative experimental consequences in all simple cases and when at the same time we have checked that the application of the theory never contains inner contradictions. For example, we believe that we understand the physical content of Einstein's concept of a closed 3-dimensional space because we can visualize consistently the experimental consequences of this concept. Of course these consequences contradict our everyday physical concepts of space and time. However, we can convince ourselves that the possibility of employing usual space-time concepts at cosmological distances can be justified neither by logic nor by observation. The physical interpretation of quantum mechanics is still full of internal discrepancies, which show themselves in arguments about continuity versus discontinuity and particle versus wave. Already from this circumstance one might conclude that no interpretation of quantum mechanics is possible which uses ordinary kinematical and mechanical concepts. Of course, quantum mechanics arose exactly out of the attempt to break with all ordinary kinematic concepts and to put in their place relations between concrete and experimentally determinable numbers. Moreover, as this enterprise seems to have succeeded, the mathematical scheme of quantum mechanics needs no revision. Equally unnecessary is a revision of space-time geometry at small distances, as we can make the quantum-mechanical laws approximate the classical ones arbitrarily closely by choosing sufficiently great masses, even when arbitrarily small distances and times come into question. But that a revision of kinematical and mechanical concepts is necessary

seems to follow directly from the basic equations of quantum mechanics. When a definite mass m is given, in our everyday physics it is perfectly understandable to speak of the position and the velocity of the center of gravity of this mass. In quantum mechanics, however, the relation $\mathbf{pq} - \mathbf{qp} = -i\hbar$ between mass, position, and velocity is believed to hold. Therefore we have good reason to become suspicious every time uncritical use is made of the words "position" and "velocity." When one admits that discontinuities are somehow typical of processes that take place in small regions and in short times, then a contradiction between the concepts of "position" and "velocity" is quite plausible. If one considers, for example, the motion of a particle in one dimension, then in continuum theory one will be able to draw (Fig. 1) a worldline $x(t)$ for the track of the particle (more precisely, its center of gravity), the tangent of which gives the velocity at every instant. In contrast, in a theory based on discontinuity there might be in place of this curve a series of points at finite separation (Fig. 2). In this case it is clearly meaningless to speak about one velocity at one position (1) because one velocity can only be defined by two positions and (2), conversely, because any one point is associated with two velocities.
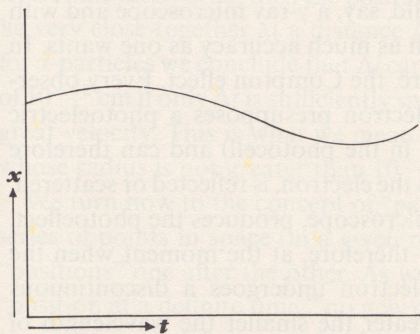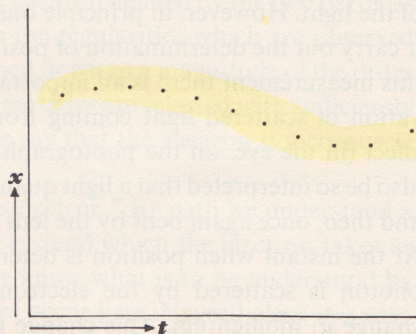

FIGURE 1   FIGURE 2

The question therefore arises whether, through a more precise analysis of these kinematic and mechanical concepts, it might be possible to clear up the contradictions evident up to now in the physical interpretations of quantum mechanics and to arrive at a physical understanding of the quantum-mechanical formulas.*

§1. CONCEPTS: POSITION, PATH, VELOCITY, ENERGY

In order to be able to follow the quantum-mechanical behavior of any object one has to know the mass of this object and its interactions with any fields and other objects. Only then can the Hamiltonian function be written down for the quantum-mechanical system. (The following considerations ordinarily refer to nonrelativistic quantum mechanics, as the laws of quantum electrodynamics are still very incompletely known.)* About the "Gestalt" (construction) of the object any further assumption is unnecessary; one most usefully employs the word "Gestalt" to designate the totality of these interactions.

When one wants to be clear about what is to be understood by the words "position of the object," for example of the electron (relative to a given frame of reference), then one must specify definite experiments with whose help one plans to measure the "position of the electron"; otherwise this word has no meaning. There is no shortage of such experiments, which in principle even allow one to determine the "position of the electron" with arbitrary accuracy. For example, let one illuminate the electron and observe it under a microscope. Then the highest attainable accuracy in the measurement of position is governed by the wavelength of the light. However, in principle one can build, say, a γ-ray microscope and with it carry out the determination of position with as much accuracy as one wants. In this measurement there is an important feature, the Compton effect. Every observation of scattered light coming from the electron presupposes a photoelectric effect (in the eye, on the photographic plate, in the photocell) and can therefore also be so interpreted that a light quantum hits the electron, is reflected or scattered, and then, once again bent by the lens of the microscope, produces the photoeffect. At the instant when position is determined—therefore, at the moment when the photon is scattered by the electron—the electron undergoes a discontinuous change in momentum. This change is the greater the smaller the wavelength of the light employed—that is, the more exact the determination of the position. At the instant at which the position of the electron is known, its momentum therefore can be known up to magnitudes which correspond to that discontinuous change. Thus, the more precisely the position is determined, the less precisely the momentum is known, and conversely. In this circumstance we see a direct physical interpretation of the equation $\mathbf{pq} - \mathbf{qp} = -i\hbar$. Let q_1 be the precision with which the value q is known (q_1 is, say, the mean error of q), therefore here the wavelength of the light. Let p_1 be the precision with which the value p is determinable; that is, here, the discontinuous change of p in the Compton effect. Then, according to the elementary laws of the Compton effect p_1 and q_1 stand in the relation

* Quite recently, however, great advances in this domain have been made in the papers of P. Dirac [*Proc. Roy. Soc. A114*, 243 (1927) and papers to appear subsequently].

$$p_1 q_1 \sim h. \tag{1}$$

That this relation (1) is a straightforward mathematical consequence of the rule $\mathbf{pq} - \mathbf{qp} = -i\hbar$ will be shown below. Here we can note that equation (1) is a precise expression for the facts which one earlier sought to describe by the division of phase space into cells of magnitude h. For the determination of the position of the electron one can also do other experiments—for example, collision experiments. A precise measurement of the position demands collisions with very fast particles, because for slow electrons the diffraction phenomena—which, according to Einstein, are consequences of de Broglie waves (as, for example, in the Ramsauer effect)—prevent a sharp specification of location. In a precise measurement of position the momentum of the electron again changes discontinuously. An elementary estimate of the precision using the formulas for de Broglie waves leads once more to relation (1).

Throughout this discussion the concept of "position of the electron" seems well enough defined, and only a word need be added about the "size" of the electron. When two very fast particles hit the electron one after the other within a very short time interval Δt, then the positions of the electron defined by the two particles lie very close together at a distance Δl. From the regularities which are observed for α-particles we conclude that Δl can be pushed down to a magnitude of the order of 10^{-12} cm if only Δt is sufficiently small and particles are selected with sufficiently great velocity. This is what we mean when we say that the electron is a corpuscle whose radius is not greater than 10^{-12} cm.

We turn now to the concept of "path of the electron." By path we understand a series of points in space (in a given reference system) which the electron takes as "positions" one after the other. As we already know what is to be understood by "position at a definite time," no new difficulties occur here. Nevertheless, it is easy to recognize that, for example, the often used expression, the "1s orbit of the electron in the hydrogen atom," from our point of view has no sense. In order to measure this 1s "path" we have to illuminate the atom with light whose wavelength is considerably shorter than 10^{-8} cm. However, a single photon of such light is enough to eject the electron completely from its "path" (so that only a single point of such a path can be defined). Therefore here the word "path" has no definable meaning. This conclusion can already be deduced, without knowledge of the recent theories, simply from the experimental possibilities.

In contrast, the contemplated measurement of position can be carried out on many atoms in a 1s state. (In principle, atoms in a given "stationary" state can be selected, for example, by the Stern-Gerlach experiment.) There must therefore exist for a definite state—for example, the 1s state—of the atom a probability function for the location of the electron which corresponds to the mean value for the classical orbit, averaged over all phases, and which can be determined through

the measurement with an arbitrary precision. According to Born,* this function is given by $\psi_{1s}(q)\bar{\psi}_{1s}(q)$ where $\psi_{1s}(q)$ designates the Schrödinger wave function belonging to the 1s state. With a view to later generalizations I should like to say— with Dirac and Jordan—that the probability is given by $S(1s, q)\bar{S}(1s, q)$, where $S(1s, q)$ designates that column of the matrix $S(E, q)$ of transformation from E to q that belongs to the energy $E = E_{1s}$.

In the fact that in quantum theory only the probability distribution of the position of the electrons can be given for a definite state, such as 1s, one can recognize, with Born and Jordan, a characteristically statistical feature of quantum theory as contrasted to classical theory. However, one can say, if one will, with Dirac, that the statistics are brought in by our experiments. For plainly *even in classical theory* only the probability of a definite position for the electron can be given as long as we do not know the phase of [the motion of the electron in] the atom. The distinction between classical and quantum mechanics consists rather in this: classically we can always think of the phase as determined through suitable experiments. In reality, however, this is impossible, because every experiment for the determination of phase perturbs or changes the atom. In a definite stationary "state" of the atom, the phases are in principle indeterminate, as one can see as a direct consequence of the familiar equations

$$\mathbf{Et} - \mathbf{tE} = -i\hbar \quad \text{or} \quad \mathbf{Jw} - \mathbf{wJ} = -i\hbar,$$

where $\mathbf{J}$ is the action variable and $\mathbf{w}$ is the angle variable.

The word "velocity" can easily be defined for an object by measurements when the motion is free of force. For example, one can illuminate the object with red light and by way of the Doppler effect in the scattered light determine the velocity of the particle. The determination of the velocity is the more exact the longer the wavelength of the light that is used, as then the change in velocity of the particle, per light quantum, by way of the Compton effect is so much less. The determination of position becomes correspondingly inexact, in agreement with equation (1). If one wants to measure the velocity of the electron in the atom at a definite instant, then, for example, one will let the nuclear charge and the forces arising

* The statistical interpretation of de Broglie waves was first formulated by A. Einstein (*Sitzungsber. d. preussische Akad. d. Wiss.*, p. 3 [1925]). This statistical feature of quantum mechanics then played an essential role in M. Born, W. Heisenberg, and P. Jordan, Quantenmechanik II (*Zeits. f. Physik, 35,* 557 [1926]), especially chapter 4, §3, and P. Jordan (*Zeits. f. Physik, 37,* 376 [1926]). It was analyzed mathematically in a seminal paper of M. Born (*Zeits. f. Physik, 38,* 803 [1926]) and used for the interpretation of collision phenomena. One finds how to base the probability picture on the theory of the transformation of matrices in the following papers: W. Heisenberg (*Zeits. f. Physik, 40,* 501 [1926]), P. Jordan (*Zeits. f. Physik, 40,* 661 [1926]), W. Pauli (remark in *Zeits. f. Physik, 41,* 81 [1927]), P. Dirac (*Proc. Roy. Soc. A113,* 621 [1926]), and P. Jordan (*Zeits. f. Physik, 40,* 809 [1926]). The statistical side of quantum mechanics is discussed more generally in P. Jordan (*Naturwiss., 15,* 105 [1927]) and M. Born (*Naturwiss., 15,* 238 [1927]).

from the other electrons suddenly be taken away, so that the motion from then on is force-free, and one will then carry out the measurement described above. As above, one can again easily convince oneself that a [momentum] function $p(t)$ cannot be defined for a given state—such as the 1s-state—of an atom. On the contrary, there is again a probability function for p in this state which according to Dirac and Jordan has the value $S(1s, p)\overline{S}(1s, p)$. Here $S(1s, p)$ again designates that column of the matrix $S(E, p)$—that transforms from $\mathbf{E}$ to $\mathbf{p}$—which belongs to $E = E_{1s}$.

Finally we come to experiments which allow one to measure the energy or the value of the action variable J. Such experiments are especially important because only with their help can we define what we mean when we speak of the discontinuous change of the energy and of J. The Franck-Hertz collision experiments allow one to base the measurement of the energy of the atom on the measurement of the energy of electrons in rectilinear motion, because of the validity of the law of conservation of energy in quantum theory. This measurement in principle can be carried out with arbitrary accuracy if only one forgoes the simultaneous determination of the position of the electron or its phase (see the determination of p, above), corresponding to the relation $\mathbf{E}t - t\mathbf{E} = -i\hbar$. The Stern-Gerlach experiment allows one to determine the magnetic or an average electric moment of the atom, and therefore to measure quantities which depend only on the action variable J. The phases remain undetermined in principle. It makes as little sense to speak of the frequency of the light wave at a definite instant as of the energy of an atom at a definite moment. Correspondingly, in the Stern-Gerlach experiment the accuracy of the energy measurement decreases as we shorten the time during which the atom is under the influence of the deflecting field.* Specifically, an upper bound is given for the deviating force through the circumstance that the potential energy of that deflecting force can at most vary inside the beam by an amount which is considerably smaller than the differences in energy of the stationary states. Only then will a determination of the energy of the stationary states be at all possible. Let E_1 be an amount of energy which satisfies this condition (E_1 also fixes the precision of the energy measurement). Then E_1/d specifies the highest allowable value for the deflecting force, if d is the breadth of the beam (measurable through the spacing of the slits employed). The angular deviation of the atomic beam is then $E_1 t_1/dp$, where we designate by t_1 the time during which the atoms are under the influence of the deflecting field, and by p the momentum of the atoms in the direction of the beam. This deflection must be of at least the same order of magnitude as the natural broadening of the beam brought about by the diffraction by the slits, if any measurement is to be possible. The diffraction angle is roughly λ/d if λ denotes the de Broglie wavelength; thus,

* In this connection see W. Pauli, *l.c.*, p. 61.

$$\lambda/d \sim E_1 t_1/dp,$$

or, as $\lambda = h/p,$

$$E_1 t_1 \sim h. \tag{2}$$

This equation corresponds to equation (1) and shows how a precise determination of energy can only be obtained at the cost of a corresponding uncertainty in the time.

§2. The Dirac-Jordan Theory

We might summarize and generalize the results of the preceding section in this statement: *All concepts which can be used in classical theory for the description of a mechanical system can also be defined exactly for atomic processes in analogy to the classical concepts.* The experiments which provide such a definition themselves suffer an indeterminacy introduced purely by the observational procedures we use when we ask of them the simultaneous determination of two canonically conjugate quantities. The magnitude of this indeterminacy is given by relation (1) (generalized to any canonically conjugate quantities whatsoever). It is natural in this respect to compare quantum theory with special relativity. According to relativity, the word "simultaneous" cannot be defined except through experiments in which the velocity of light enters in an essential way. If there existed a "sharper" definition of simultaneity, as, for example, signals which propagate infinitely fast, then relativity theory would be impossible. However, because there are no such signals, or, rather, because already in the definition of simultaneity the velocity of light appears, there is room left for the postulate of the constancy of the speed of light so that this postulate does not contradict any meaningful use of the words "position, velocity, time." We find a similar situation with the definition of the concepts of "position of an electron" and "velocity" in quantum theory. All experiments which we can use for the definition of these terms necessarily contain the uncertainty implied by equation (1), even though they permit one to define exactly the concepts p and q taken in isolation. If there existed experiments which allowed simultaneously a "sharper" determination of p and q than equation (1) permits, then quantum mechanics would be impossible. Thus only the uncertainty which is specified by equation (1) creates room for the validity of the relations which find their most pregnant expression in the quantum-mechanical commutation relations,

$$\mathbf{pq} - \mathbf{qp} = -i\hbar.$$

That uncertainty makes possible this equation without requiring that the physical meaning of the quantities p and q be changed.

For those physical phenomena whose quantum-mechanical formulation is still

unknown (for example, electrodynamics), equation (1) makes a demand which may be useful for the discovery of the new laws. For quantum mechanics equation (1) can be derived from the Dirac-Jordan formulation by a slight generalization. If, for any definite state variable η we determine the position q of the electron as q' with an uncertainty q_1, then we can express this fact by a probability amplitude $S(\eta, q)$ which differs appreciably from zero only in a region of spread q_1 near q'. For example, one can write

$$S(\eta, q) \text{ proportional to } \exp\left[-(q - q')^2/2q_1^2 - (ip'/\hbar)(q - q')\right], \tag{3a}$$

with therefore

$$S\bar{S}' \text{ proportional to } \exp\left[-(q - q')^2/q_1^2\right]. \tag{3b}$$

Then for the probability amplitude for any given value of p we have

$$S(\eta, p) = \int S(\eta, q)S(q, p)\, dq. \tag{4}$$

For $S(q, p)$, according to Jordan, we can write

$$S(q, p) = \exp(ipq/\hbar). \tag{5}$$

Then, according to (4), $S(\eta, p)$ differs appreciably from zero only for values of p for which $(p - p')q_1/\hbar$ is not significantly greater than 1. Specifically, employing (3), we find $S(\eta, p)$ is proportional to

$$\int \exp\left[i(p - p')q/\hbar - (q - q')^2/2q_1^2\right] dq;$$

that is, proportional to

$$\exp\left[-(p - p')^2/2p_1^2 + iq'(p - p')/\hbar\right];$$

and thus

$$S\bar{S} \text{ is proportional to } \exp\left[-(p - p')^2/p_1^2\right],$$

where

$$p_1q_1 = \hbar. \tag{6}$$

The assumption (3) for $S(\eta, q)$ corresponds therefore to the experimental fact that the value p' is measured for p and the value q' for q [with the limit (6) on the precision].

From the purely mathematical point of view it is characteristic of the Dirac-

Jordan formulation of quantum mechanics that the relations between **p**, **q**, **E**, etc. can be described as equations connecting very general matrices in such a way that any predetermined quantum-theoretic quantity appears as a diagonal matrix. The possibility of writing things in such a way is evident when one pictures the matrices as tensors (for example, moment-of-inertia tensors) in a multidimensional space between which there are mathematical connections. One can always pick the axis of the coordinate system in which one expresses these relations along the principal axes of one of these tensors. Finally, one also can always characterize the mathematical relation between two tensors A and B through the transformation equations which take a coordinate system oriented along the principal axes of A over into another oriented along the principal axes of B. This latter formulation corresponds to the Schrödinger theory. In contrast, one will view Dirac's q-number formulation as the formulation of quantum mechanics that is really "invariant" and independent of all coordinate systems. When we want to derive physical results from that mathematical framework, then we have to associate numbers with the quantum-theoretical magnitudes—that is, with the matrices (or "tensors" in multidimensional space). This task is to be understood in these terms: In that multidimensional space a definite direction is arbitrarily prescribed (by the nature of the experimental setup); and it is asked what is the "value" of the matrix (for example, in that picture, what is the value of the moment of inertia) in this given direction. This question only has a well-defined meaning when the given direction coincides with the direction of one of the principal axes of that matrix. In this case there is an exact answer for the question. But also when the prescribed direction differs only little from one of the principal axes of the matrix one can still speak of the "value" of the matrix in the prescribed direction up to a definite uncertainty determined by the angle between the two directions. One can therefore say that associated with every quantum-theoretical quantity or matrix is a number which gives its "value" within a certain definite statistical error. The statistical error depends on the coordinate system. For every quantum-theoretical quantity there exists a coordinate system in which the statistical error for this quantity is zero. Therefore a definite experiment can never give exact information on all quantum-theoretical quantities. Rather, it divides physical quantities into "known" and "unknown" (or more and less accurately known quantities) in a way characteristic of the experiment in question. The results of two experiments can be derived exactly one from the other only then when the two experiments divide the physical quantities in the same way into "known" and "unknown" (that is, when the tensors in that multidimensional space frequently invoked—for ease of visualization—are "looked at" in both experiments from the same direction). When two experiments use different divisions into "known" and "unknown," then their results can be related only statistically.

For a more detailed discussion of this statistical connection let a *gedanken-experiment* be considered. Let a Stern-Gerlach atomic beam be sent first through

a field F_1 which is so strongly inhomogeneous in the direction of the beam that it induces many transitions by sudden reversal in the force on the spin. Then let the atomic beam run free up to a definite distance from F_1. But there let a second field F_2 begin, as inhomogeneous as F_1. Between F_1 and F_2 let it be possible to measure the number of atoms in the different stationary states through an optionally applied magnetic field. Let all radiation by the atoms be neglected. If we know that an atom was in a state of energy E_n before it passed F_1, then we can express this fact by ascribing to the atom a wave function—for example, in p-space—with the definite energy E_n and the undetermined phase β_n,

$$S(E_n, p) = \psi(E_n, p) \exp[-i(E_n/\hbar)(\alpha + \beta_n)].$$

After passage through the field F_1, this function is changed into*

$$S(E_n, p) \xrightarrow{F_1} \sum_m c_{nm}\psi(E_m, p) \exp[-i(E_m/\hbar)(\alpha + \beta_m)]. \qquad (7)$$

Here we can make some arbitrary determination of the β_m so that the c_{nm} are uniquely determined by F_1. The matrix c_{nm} transforms the energy values before the transition through F_1 to the values after the transition. If after F_1 we carry out a determination of the stationary state, say, by use of an inhomogeneous magnetic field, then we will find that the atom has jumped from the nth state to the mth state with a probability $c_{nm}\bar{c}_{nm}$. When we find experimentally that an atom has indeed jumped to the mth state, then we have to ascribe to it in all calculations thereafter, not the function $\sum_m c_{nm}S_m$, but simply the function S_m with an undetermined phase. Through the experimental determination, "mth state," we select out of the multitude of different possibilities (c_{nm}) a definite one, m. However, at the same time we disturb everything that was still contained in the phase relations between the quantities c_{nm}, as detailed below. In the transition of the atomic beam through F_2, what happened at F_1 repeats itself. Let d_{nm} be the coefficients of the transformation matrix which transform the energies before F_2 to the energies after F_2. If no determination of the state is carried out between F_1 and F_2, then the eigenfunction is transformed according to the following scheme,

$$S(E_n, p) \xrightarrow{F_1} \sum_m c_{nm}S(E_m, p) \xrightarrow{F_2} \sum_m \sum_l c_{nm}d_{ml}S(E_l, p). \qquad (8)$$

Let $\sum_m c_{nm}d_{ml}$ be called e_{nl}. If the stationary state of the atom is determined beyond F_2, then one will find the state l with the probability $e_{nl}\bar{e}_{nl}$. In contrast, if between F_1 and F_2 one determines the state—and finds for it the value E_m—then

* See P. Dirac (*Proc. Roy. Soc. A112*, 661 [1926]) and M. Born (*Zeits. f. Physik, 40*, 167 [1926]).

the probability for "l" beyond F_2 is given by $d_{ml}\bar{d}_{ml}$. In many repetitions of the entire experiment (in which each time the state is determined between F_1 and F_2) one will therefore observe the state l, beyond F_2, with the relative frequency $Z_{nl} = \sum_m c_{nm}\bar{c}_{nm}d_{ml}\bar{d}_{ml}$. This expression does not agree at all with $e_{nl}\bar{e}_{nl}$. For this reason Jordan (*l.c.*) has spoken of an "interference of probabilities." I cannot agree. The two kinds of experiments which lead respectively to $e_{nl}\bar{e}_{nl}$ and Z_{nl} are physically distinct. In one case the atom experiences no disturbance between F_1 and F_2. In the other case it is perturbed by the apparatus which determines its stationary state. This apparatus has as a consequence that the "phase" of the atom changes by an amount that is in principle uncheckable, as the momentum of an electron likewise changes with a determination of its position (see §1). The magnetic field for the determination of the state between F_1 and F_2 will separate the eigenvalues E. In the observation of the path of the atomic beam the atoms are slowed down by statistically different and uncheckable amounts (I think here, say, of Wilson cloud-chamber pictures). This has as a consequence that the final transformation matrix (from the energy value before entry into F_1 to the energy after exit from F_2) is no longer given by $\sum_m c_{nm}d_{ml}$, but every term in this sum has additionally an unknown phase factor. No expectation is therefore open to us, except that the mean value of $e_{nl}\bar{e}_{nl}$ averaged over all these expected phase alterations is equal to Z_{nl}. A simple calculation confirms that this is the case. We can therefore deduce from one experiment the possible results of another by definite statistical rules. The other experiment itself selects out of the plenitude of all possibilities a quite definite one, and thereby limits the possibilities for all later experiments. Such an interpretation of the equation for the transformation matrix S or the Schrödinger wave equation is only possible because the sum of solutions is again a solution. In this circumstance we see the deep significance of the linearity of Schrödinger wave equations. On that account they can be understood only as equations for waves in phase space; and on that account we may regard as hopeless every attempt to replace these equations by nonlinear equations, for example in the relativistic case (for more than one electron).

§3. The Transition from Micro- to Macromechanics

It seems to me that the concepts of kinematics and mechanics in quantum theory are sufficiently clarified by the analysis of the words "position of an electron," "velocity," "energy," etc., in the preceding sections that physical understanding of macroscopic processes from the standpoint of quantum mechanics must also be possible. The transition from micro- to macromechanics has already been treated by Schrödinger,* but I do not believe that Schrödinger's considerations

* E. Schrödinger, *Naturwiss.*, *14*, 664 (1926).

get to the heart of the problem, and this is why: According to Schrödinger, in a high state of excitation a sum of eigenfunctions ought to be able to give a wave packet of limited extent which—through periodic changes in its form—will carry out the periodic motions of the classical "electron." There is an argument against this outlook: If the wave packet had such properties as ascribed to it by this view, then the radiation sent out by the atom would be representable as a Fourier series in which the frequencies of the higher vibrations were integer multiples of the basic frequency. The frequencies of the spectral lines sent out by the atom are, however, according to quantum mechanics, never integer multiples of the basic frequency—except in the special case of the harmonic oscillator. Thus Schrödinger's reasoning is only viable for the case of the harmonic oscillator treated by him; in all other cases a wave packet spreads out in the course of time over the whole immediate neighborhood of the atom. The higher the state of excitation of the atom, the slower is that spreading of the wave packet. However, if one waits long enough it happens. The reasoning developed above about the radiation sent out by the atom might at first sight be used against all experiments which look for a direct transition from quantum to classical mechanics at high quantum numbers. For that reason the attempt was made earlier to circumvent such reasoning by referring to the natural radiation broadening of stationary states—certainly wrongly, first of all because this way out is blocked already in the case of the hydrogen atom on account of the weakness of the radiation for high states, and secondly, because the transition from quantum to classical mechanics ought to be understandable without calling on electrodynamics. Bohr* has already referred many times to these well-known difficulties which stand in the way of a direct connection between quantum and classical theory. We have spelled them out again here so explicitly only because in recent times they seem to be forgotten.

I believe that one can fruitfully formulate the origin of the classical "orbit" in this way: the "orbit" comes into being only when we observe it. For example, let an atom be given in a state of excitation $n = 1000$. The dimensions of the orbit in this case are already relatively large so that, in accordance with §1, it is enough to use light of relatively low wavelength to determine the position of the electron. If the position determination is not to be too fuzzy then the Compton recoil will put the atom in some state of excitation between, say, 950 and 1050. Simultaneously, the momentum of the electron can be determined from the Doppler effect with a precision given by (1). One can characterize the experimental finding by a wave-packet, or, better, a probability-amplitude packet, in q-space of a spread given by the wavelength of the light used, and built up primarily out of eigenfunctions between the 950th and 1050th eigenfunction—and by a corresponding packet in p-space. Let a new determination of position be made after some time with the same precision. Its result, according to §2, can be predicted only statistically. All

* N. Bohr, "Grundpostulate der Quantentheorie," *l.c.*

positions count as likely (with calculable probability) which lie within the bounds of the now broadened wavepacket. The situation would be no different in classical theory, for there too the result of the second position measurement would be predictable only statistically because of the uncertainty in the first measurement. Also, the orbits of classical theory would spread out like the wavepacket. However, statistical laws themselves are different in quantum mechanics and in classical theory. The second determination of the position selects a definite "q" from the totality of possibilities and limits the options for all subsequent measurements. After the second position determination the results of later measurements can only be calculated when one again ascribes to the electron a "smaller" wavepacket of extension λ (wavelength of the light used in the observation). Thus every position determination reduces the wavepacket back to its original extension λ. The "values" of the quantities $\mathbf{p}$, $\mathbf{q}$ are known throughout all the experiments with a certain precision. The values of $\mathbf{p}$ and $\mathbf{q}$ stay within the precision limits fixed by the classical equations of motion. This can be seen directly from the quantum-mechanical equations,

$$\dot{\mathbf{p}} = -\partial \mathbf{H}/\partial \mathbf{q}; \qquad \dot{\mathbf{q}} = \partial \mathbf{H}/\partial \mathbf{p}. \tag{9}$$

However, the orbit, as noted earlier, can be calculated only statistically from the initial conditions, a circumstance that one can consider as a consequence of the fundamental indeterminism of the initial conditions. The statistical laws are different for quantum mechanics and classical theory. This distinction, under appropriate circumstances, can give rise to gross macroscopic differences between classical and quantum theory. Before I discuss an example, I should like to indicate how the transition, discussed above, to classical theory is formulated mathematically for a simple mechanical system, the force-free motion of a particle. In one dimension the equations of motion run

$$\mathbf{H} = \mathbf{p}^2/2m; \qquad \dot{\mathbf{q}} = \mathbf{p}/m; \qquad \dot{\mathbf{p}} = 0. \tag{10}$$

As time can be treated as a parameter (or "c-number") when there are no time-dependent external forces, the solution of this equation is

$$\mathbf{q} = \mathbf{p}_0 t/m + \mathbf{q}_0; \qquad \mathbf{p} = \mathbf{p}_0, \tag{11}$$

where $\mathbf{p}_0$ and $\mathbf{q}_0$ are the momentum and the position at the time $t = 0$. At $t = 0$ the value $q_0 = q'$ is measured with accuracy q_1, and $p_0 = p'$ with accuracy p_1 [see equations (3) to (6)]. In order to draw conclusions from the "values" of $\mathbf{p}_0$ and $\mathbf{q}_0$ about the "values" of $\mathbf{q}$ at the time t, one must find—according to Dirac and Jordan—that transformation function which transforms all matrices in the

representation* in which $\mathbf{q}_0$ is diagonal to that representation in which $\mathbf{q}$ is diagonal. In the matrix scheme in which $\mathbf{q}_0$ is diagonal, $\mathbf{p}_0$ can be replaced by the operator $-i\hbar \, \partial/\partial q_0$. According to Dirac [*l.c.* equation (11)] the desired transformation amplitude $S(q_0, q)$ satisfies the differential equation,

$$\{-i(\hbar t/m)\,\partial/\partial q_0 + q_0\}S(q_0, q) = qS(q_0, q) \tag{12}$$

or

$$-i(\hbar t/m)\,\partial S/\partial q_0 = (q_0 - q)S(q_0, q),$$

[with the solution]

$$S(q_0, q) = \text{const} \exp\left[(im/\hbar t)\int(q - q_0)\,dq_0\right]. \tag{13}$$

$S\bar{S}$ is therefore independent of q_0. In other words, if at the time $t = 0$ we know q_0 exactly, then at any time $t > 0$ all values of q are equally probable; that is, the probability that q lies in any finite region is quite nil. Physically this is already clear without further investigation. Thus the exact determination of q_0 leads to an infinitely large Compton recoil. The same would naturally apply for an arbitrary mechanical system. If, however, q_0 is known at the time $t = 0$ only within the range q_1, and p_0 in the range p_1 [see equation (3)], then

$$S(\eta, q_0) = \text{const} \exp\left[-(q_0 - q')^2/2q_1^2 - (i/h)p'(q_0 - q')\right],$$

and the probability [amplitude] function for q is to be calculated according to the formula,

$$S(\eta, q) = \int S(\eta, q_0)S(q_0, q)\,dq_0.$$

The result is

$$S(\eta, q) = \text{const} \int \exp\{(im/\hbar t)[q_0(q - tp'/m) - q_0^2/2] - (q' - q_0)^2/2q_1^2\}\,dq_0. \tag{14}$$

With the abbreviation

$$\beta = th/mq_1^2, \tag{15}$$

the exponent in (14) becomes

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* The word "representation," not employed by Heisenberg himself, is introduced here for clarity. He uses the phrase "matrix scheme." (Translators' note.)

$$-\{q_0^2(1 + i/\beta) - 2q_0[q' + i(q - tp'/m)/\beta] + q'^2\}/2q_1^2.$$

The term with q'^2 can be taken into the constant (q-independent factor) and the integration gives

$$S(\eta, q) = \text{const} \exp\{[q' + i(q - tp'/m)/\beta]^2/[2(1 + i/\beta)q_1^2]\}$$
$$= \text{const} \exp - \{[(q - tp'/m - i\beta q')^2(1 - i/\beta)]/[2q_1^2(1 + \beta^2)]\}. \quad (16)$$

It follows that

$$S(\eta, q)\overline{S}(\eta, q) = \text{const} \exp - \{[q - tp'/m - q']^2/q_1^2(1 + \beta^2)\}. \quad (17)$$

Thus the electron is located at the time t at the position $tp'/m + q'$ with a spread $q_1(1 + \beta^2)^{1/2}$. The "wavepacket" or, better, "probability packet" has expanded by the factor $(1 + \beta^2)^{1/2}$. According to (15), β is proportional to the time t, inversely proportional to the mass, as is entirely plausible, and inversely proportional to q_1^2. Too much precision in q_0 produces great uncertainty in p_0 and thus leads to a large uncertainty in q. The parameter η which we have brought in above for formal reasons might be left out here in all formulas, as it does not enter the calculation. To illustrate that the difference between the classical and the quantum statistical laws leads to gross macroscopic differences between the results of the two theories, let the reflection of a beam of electrons at a grating be discussed briefly. When the spacing of the rulings is of the order of the de Broglie wavelength of the electrons, then reflection occurs in definite, discrete directions like the reflection of light at a grating. What classical theory gives is grossly and macroscopically different. Nevertheless, from the orbit of an individual electron we can in no way find a contradiction with a classical theory. We might if we could, direct the electron, say, to a definite point on a grating ruling, and then verify that the reflection there violates classical theory. However, when we want to determine the position of the electron so precisely that we can say at what point on a grating ruling it hits, then the electron acquires through this position determination a large velocity, and the de Broglie wavelength of the electron becomes so much shorter that now the reflection really can and will take place approximately as predicted classically, without violating the laws of the quantum theory.

§4. Discussion of a Few Special Idealized Experiments

According to the physical interpretation of quantum theory aimed at here, the time of transitions or "quantum jumps" must be as concrete and determinable by measurement as, say, energies in stationary states. The spread within which such an instant is specifiable is given according to equation (2) by $h/\Delta E$, if ΔE designates

the change of energy in a quantum jump.* We consider, for example, the following experiment. An atom, at time $t = 0$ in state $n = 2$ may transit, via radiation, to the ground state, $n = 1$. Then, in analogy to equation (7), we can ascribe to the atom the eigenfunction

$$S(t, p) = e^{-\alpha t}\psi(E_2, p)e^{-iE_2 t/\hbar} + (1 - e^{-2\alpha t})^{1/2}\psi(E_1, p)e^{-iE_1 t/\hbar} \qquad (18)$$

if we assume that the radiation damping is expressed in a factor of the form $e^{-\alpha t}$ in the eigenfunction (the real dependence is perhaps not so simple). This atom is sent through an inhomogeneous magnetic field for the determination of its energy level, as is usual in the Stern-Gerlach experiment; yet we also have the inhomogeneous field follow the atomic beam over a long stretch of its path. The consequent acceleration we will measure, say, in this way: we divide the entire stretch that the atomic beam pursues in the magnetic field into short sections, at the end of each of which we determine the deviation of the beam. Depending on the velocity of the atomic beam, the division into intervals of space corresponds for the atom to division into small time intervals Δt. According to §1, equation (2), there corresponds to a time interval Δt a spread in energy of $h/\Delta t$. The probability of measuring the definite energy E can be directly deduced from $S(p, E)$ and is therefore calculated for the interval from $n\,\Delta t$ to $(n + 1)\,\Delta t$ as,

$$S(p, E) \Big|_{n\,\Delta t \to (n+1)\,\Delta t} = \int_{n\,\Delta t}^{(n+1)\,\Delta t} S(p, t)e^{iEt/\hbar}\, dt.$$

If the determination "state $n = 2$" is made at the time $(n + 1)\,\Delta t$, then for everything later one must ascribe to the atom not the eigenfunction (18) but one which results from (18) when t is replaced by $t - (n + 1)\,\Delta t$. If, on the contrary, one finds "state $n = 1$," then from that point on one has to attribute to the atom the eigenfunction

$$\psi(E_1, p)e^{-iE_1 t/\hbar}.$$

Thus one will first find for a series of intervals Δt "state $n = 2$," then steadily "state $n = 1$." In order that a distinction between the two states will still be possible, Δt cannot be shrunk below $h/\Delta E$. Thus the instant of the transition is determinable within this spread. We imply an experiment of the kind just sketched quite in the spirit of the old formulation of quantum theory founded by Planck, Einstein, and Bohr when we speak of the discontinuous change of the energy. As such an experiment can in principle be carried out, an agreement about its outcome must be possible.

In Bohr's basic postulates of quantum theory, the energy of an atom has the

* Compare W. Pauli, *l.c.*, p. 12.

advantage—just as do the values of the action variables J—over other deter-
minants of the motion (position of the electron, etc.) in that its numerical value
can always be given. This preferred position which the energy has over other
quantum-mechanical quantities it owes only to the circumstance that it represents
an integral of the equations of motion for closed systems (the energy matrix $\mathbf{E}$ is
a constant). For open systems, in contrast, the energy is not singled out over any
other quantum-mechanical quantity. In particular, one will be able to devise
experiments in which the phases, w, of the atom are precisely measurable, but in
which then the energy remains in principle undetermined, corresponding to the
relation $\mathbf{J}\mathbf{w} - \mathbf{w}\mathbf{J} = -i\hbar$ or $J_1 w_1 \sim h$. Resonance fluorescence is such an experi-
ment. If one irradiates an atom with an eigenfrequency, say $v_{12} = (E_2 - E_1)/h$,
then the atom vibrates in phase with the external radiation. Then, even in principle,
it makes no sense to ask in which state, E_1 or E_2, the atom is thus vibrating. The
phase relation between atom and external radiation may be determined, for
example, by the phase relations of large numbers of atoms with one another
([R. W.] Wood's experiments). If one prefers to avoid experiments with radiation
then one can also measure the phase relation by carrying out exact position
determinations on the electron in the sense of §1 at different times relative to the
phase of the light impinging (on many atoms). A "wave function," say, of the form,

$$S(q, t) = c_2 \psi_2(E_2, q) e^{-i(E_2 t + \beta)/\hbar} + (1 - c_2^2)^{1/2} \psi_1(E_1, q) e^{-iE_1 t/\hbar}, \qquad (19)$$

can be ascribed to the individual atom. Here c_2 depends on the strength and β on
the phase of the incident light. The probability of a definite position q is thus

$$S(q, t)\overline{S}(q, t) = c_2^2 \psi_2 \overline{\psi}_2 + (1 - c_2^2)\psi_1 \overline{\psi}_1$$
$$+ c_2(1 - c_2^2)^{1/2}\{\psi_2\overline{\psi}_1 e^{-i[(E_2 - E_1)t + \beta]/\hbar} + \overline{\psi}_2\psi_1 e^{+i[(E_2 - E_1)t + \beta]/\hbar}\}. \qquad (20)$$

The periodic term in (20) is experimentally distinguishable from the unperiodic
ones, as the determinations of position can be carried out for different phases of
the incident light.

   In a well-known idealized experiment proposed by Bohr, the atoms of a Stern-
Gerlach atomic beam are first excited to a resonance fluorescence at a definite
state by incident radiation. After a little way they go through an inhomogeneous
magnetic field. The radiation emerging from the atoms can be observed during the
whole path, before and after the magnetic field. Before the atoms enter the magnetic
field, ordinary resonance radiation takes place; that is, as in dispersion theory, it
must be assumed that all atoms send out spherical waves in phase with the incident
light. The latter view at first sight contradicts the result that a crude application
of the quantum theory of light or the basic rules of quantum theory would give.
Thus from this view one would conclude that only a few atoms are raised to the

"upper state" through absorption of the light quantum, and that therefore the entire resonance radiation arises from a few intensively radiating centers. It therefore seemed natural in earlier times to say that the concept of light quanta ought to be brought in here only to account for the balance of energy and momentum, and that "in reality" all atoms in the ground state radiate weak and coherent spherical waves. After the atoms have passed the magnetic field, however, there can hardly be any doubt that the atomic beam has divided into two beams of which one corresponds to atoms in the upper state, the other in the lower. If now the atoms in the lower state were to radiate, then we would have a gross violation of the law of conservation of energy. For all the energy of excitation resides in the beam with atoms in the upper state. Still less can there be any doubt that past the magnetic field only the "upper state" beam sends out light, and indeed incoherent light, from the few intensively radiating atoms in the upper state. As Bohr has shown, this idealized experiment makes it especially clear that care is often needed in applying the concept of "stationary state." The formulation of quantum theory developed here allows a discussion of the Bohr experiment to be carried through without any difficulty. In the external radiation field the phases of the atoms are determined. Therefore it is meaningless to speak of the "energy of the atom." Also, after the atom has left the radiation field one is not entitled to say that it is in a definite stationary state, insofar as one enquires about the coherence properties of the radiation. However, one can set up an experiment to find out in what state the atom is. The result of this experiment can be stated only statistically. Such an experiment is really performed by the inhomogeneous magnetic field. Beyond the magnetic field the energies of the atoms are well determined and therefore the phases are indeterminate. The resulting radiation is incoherent and comes only from atoms in the upper state. The magnetic field determines the energies and therefore destroys the phase relation. Bohr's idealized experiment is a very beautiful illustration of the fact that the energy of the atom "in reality" is not a number but a matrix. The conservation law holds for the energy matrix and therefore also for the value of the energy as precisely as it can be measured. In mathematical terms the lifting of the phase relation can be traced out as follows, for example. Let Q be the coordinates of the center of gravity of the atom, so that one ascribes to the atom, instead of (19), the eigenfunction

$$S(Q, t)S(q, t) = S(Q, q, t). \tag{21}$$

Here $S(Q, t)$ is a function that, like $S(\eta, q)$ in (16), differs from zero only in a small neighborhood of a point in Q-space and propagates with the velocity of the atoms in the direction of the beam. The probability of a relative amplitude q regardless of Q is given by the integral of

$$S(Q, q, t)\bar{S}(Q, q, t)$$

over Q—that is, through (20). However, the eigenfunction (21) will change by a calculable amount in a magnetic field and, on account of the different deviation of atoms in the upper and lower state, will have changed beyond the magnetic field into

$$S(Q, q, t) = c_2 S_2(Q, t)\psi_2(E_2, q)e^{-i(E_2 t + \beta)/\hbar} + (1 - c_2^2)^{1/2}S_1(Q, t)\psi_1(E_1, q)e^{-iE_1 t/\hbar}. \quad (22)$$

Here $S_1(Q, t)$ and $S_2(Q, t)$ will be functions in Q-space which differ from zero only in the small neighborhood of a point; but this point is different for S_1 and S_2. The product $S_1 S_2$ is therefore zero everywhere. The probability of a relative amplitude q and a definite value Q is therefore

$$S(Q, q, t)\bar{S}(Q, q, t) = c_2^2 S_2 \bar{S}_2 \psi_2 \bar{\psi}_2 + (1 - c_2^2)S_1 \bar{S}_1 \psi_1 \bar{\psi}_1. \quad (23)$$

The periodic term of (20) has disappeared and with it the possibility of measuring a phase relation. The statistical result of position determinations will always be the same, whatever the phase of the incident radiation. We may assume that experiments with radiation, the theory of which has not yet been developed, will give the same results about phase relations between atoms and incident radiation.

Finally let us examine the connection* of equation (2), $E_1 t_1 \sim h$, with a complex of problems which Ehrenfest and other investigators have discussed on the basis of Bohr's correspondence principle in two important papers.† Ehrenfest and Tolman speak of "weak quantization" when the quantized periodic motion is interrupted through quantum jumps or rather perturbations in intervals of time which can be regarded as not very long compared to the periods of the system. These cases should reveal not only the exact quantum energy values but also energy values which do not differ too much from the quantum values, and these with a smaller and qualitatively predictable *a priori* probability. In quantum mechanics this behavior is to be interpreted in these terms. As the energy is really changed by external perturbations or quantum jumps, every energy measurement, insofar as it is to be unique, must be done in the time between two perturbations. In this way an upper bound is specified for t_1 in the sense of §1. Therefore we measure the energy value E_0 of a quantized state also only within a spread $E_1 \sim h/t_1$. Here the question is meaningless even in principle whether the system "really" takes on with the correspondingly lower statistical weight such energy values E as deviate from E_0, or whether their experimental realization is to be attributed only to the inaccuracy of the measurement. If t_1 is smaller than the period of the system then it is no longer meaningful to speak of discrete stationary states or discrete energy values.

* W. Pauli drew my attention to this connection.
† P. Ehrenfest and G. Breit (*Zeits. f. Physik*, 9, 207 [1922]) and P. Ehrenfest and R. C. Tolman (*Phys. Rev.*, 24, 287 [1924]). See also the discussion in N. Bohr, *Grundpostulate der Quantentheorie, l.c.*

Ehrenfest and Breit in a similar connection draw attention to the following paradox. A rotator, which we will visualize as a gear-wheel, is provided with an attachment which after f revolutions of the wheel exactly reverses the direction of its rotation. For example, let the gear-wheel mesh with a toothed sliding member which moves on a straight line between two stops. The slider hits a stop after a definite number of rotations and in that way reverses the rotation of the gear-wheel. The true period T of the system is long in comparison with the rotation period t of the wheel. The discrete energy levels are densely packed—and the denser the packing, the greater the value of T. From the standpoint of consistent quantum theory all stationary states have the same statistical weight. Therefore, for sufficiently great T, practically all energy values occur with equal frequency, in opposition to what would be expected for the rotator. We may sharpen this paradox a little before we treat it from our standpoint. Thus, in order to determine whether the system takes on the discrete energy values belonging to the pure rotator exclusively or particularly often, or whether it assumes with equal probability all possible values (that is, values which correspond to the small energy interval h/T), a time t_1 suffices which is small relative to T (but $\gg t$). In other words, although the long period plays no part at all in such measurements, it appears to express itself in the fact that all possible energy values can occur. We are of the view that, in reality also, such experiments for the determination of the total energy of the system would give all possible energy values with equal probability. The factor responsible for this outcome is not the big period T, but the sliding member. Even if the system sometimes happens to have an energy identical with the quantized energy value of the simple rotator, it can be modified easily—by external forces acting on the stop—to states which do not correspond to the quantization of the simple rotator.* The coupled system, rotator-plus-slider, indeed shows a periodicity entirely different from that of the rotator. The solution of the paradox lies rather in a different circumstance. When we want to measure the energy of the rotator alone, we must first break the coupling between the rotator and the slider. In classical theory, when the mass of the slider is sufficiently small, the coupling can be broken without a change in energy; and there, consequently, the energy of the entire system can be equated to that of the rotator (for small slider mass). In quantum mechanics the energy of interaction between slider and rotator is at least of the same order of magnitude as the level spacing of the rotator (even for small slider mass there is a high zero point energy associated with the elastic interaction between rotator and slider). On decoupling, the slider and the rotator individually take their characteristic quantum energies. Consequently, insofar as we are able to measure the energy values of the rotator alone we always find the quantum energy values with experimental accuracy. Even for

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* According to Ehrenfest and Breit this cannot happen, or can happen only rarely, through forces which act on the gear-wheel.

vanishing mass of the slider, however, the energy of the coupled system is different from the energy of the rotator. The energy of the coupled system can take on all possible values (consistent with the T-quantization) with equal probability.

Quantum kinematics and mechanics show far-reaching differences from the ordinary theory. The applicability of classical kinematics and mechanical concepts, however, can be justified neither from our laws of thought nor from experiment. The basis for this conclusion is relation (1), $p_1 q_1 \sim h$. As momentum, position, energy, etc. are precisely defined concepts, one does not need to complain that the basic equation (1) contains only qualitative predictions. Moreover, as we can think through qualitatively the experimental consequences of the theory in all simple cases, we will no longer have to look at quantum mechanics as unphysical and abstract.* Of course we would also like to be able to derive, if possible, the quantitative laws of quantum mechanics directly from the physical foundations—that is, essentially, from relation (1). On this account Jordan has sought to interpret the equation,

$$S(q, q'') = \int S(q, q') S(q', q'') \, dq',$$

as a probability relation. However, we cannot accept this interpretation (§2). We believe, rather, that for the time being the quantitative laws can be derived out of the physical foundations only by use of the principle of maximum simplicity. If, for example, the X-coordinate of the electron is no longer a "number," as can be concluded experimentally, according to equation (1), then the simplest assumption conceivable [that does not contradict (1)] is that this X-coordinate is a diagonal term of a matrix whose nondiagonal terms express themselves in an uncertainty or—by transformation—in other ways (see for example §4). The prediction that, say, the velocity in the X-direction is "in reality" not a number but the diagonal term of the matrix, is perhaps no more abstract and no more unvisualizable than the statement that the electric field strengths are "in reality" the time part of an antisymmetric tensor located in space-time. The phrase "in reality" here is as much and as little justified as it is in any mathematical description of natural processes. As soon as one accepts that all quantum-theoretical quantities are "in reality" matrices, the quantitative laws follow without difficulty.

If one assumes that the interpretation of quantum mechanics is already correct

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* Schrödinger describes quantum mechanics as a formal theory of frightening, indeed repulsive, abstractness and lack of visualizability. Certainly one cannot overestimate the value of the mathematical (and to that extent physical) mastery of the quantum-mechanical laws that Schrödinger's theory has made possible. However, as regards questions of physical interpretation and principle, the popular view of wave mechanics, as I see it, has actually deflected us from exactly those roads which were pointed out by the papers of Einstein and de Broglie on the one hand and by the papers of Bohr and by quantum mechanics on the other hand.

in its essential points, it may be permissible to outline briefly its consequences of principle. We have not assumed that quantum theory—in opposition to classical theory—is an essentially statistical theory in the sense that only statistical conclusions can be drawn from precise initial data. The well-known experiments of Geiger and Bothe, for example, speak directly against such an assumption. Rather, in all cases in which relations exist in classical theory between quantities which are really all exactly measurable, the corresponding exact relations also hold in quantum theory (laws of conservation of momentum and energy). But what is wrong in the sharp formulation of the law of causality, "When we know the present precisely, we can predict the future," is not the conclusion but the assumption. Even in principle we cannot know the present in all detail. For that reason everything observed is a selection from a plenitude of possibilities and a limitation on what is possible in the future. As the statistical character of quantum theory is so closely linked to the inexactness of all perceptions, one might be led to the presumption that behind the perceived statistical world there still hides a "real" world in which causality holds. But such speculations seem to us, to say it explicitly, fruitless and senseless. Physics ought to describe only the correlation of observations. One can express the true state of affairs better in this way: Because all experiments are subject to the laws of quantum mechanics, and therefore to equation (1), it follows that quantum mechanics establishes the final failure of causality.

ADDITION IN PROOF

After the conclusion of the foregoing paper, more recent investigations of Bohr have led to a point of view which permits an essential deepening and sharpening of the analysis of quantum-mechanical correlations attempted in this work. In this connection Bohr has brought to my attention that I have overlooked essential points in the course of several discussions in this paper. Above all, the uncertainty in our observation does not arise exclusively from the occurrence of discontinuities, but is tied directly to the demand that we ascribe equal validity to the quite different experiments which show up in the corpuscular theory on one hand, and in the wave theory on the other hand. In the use of an idealized gamma-ray microscope, for example, the necessary divergence of the bundle of rays must be taken into account. This has as one consequence that in the observation of the position of the electron the direction of the Compton recoil is only known with a spread which then leads to relation (1). Furthermore, it is not sufficiently stressed that the simple theory of the Compton effect, strictly speaking, only applies to free electrons. The consequent care needed in employing the uncertainty relation is, as Professor Bohr has explained, essential, among other things, for a comprehensive discussion of the transition from micro- to macromechanics. Finally, the discussion of resonance fluorescence is not entirely correct because the connection between

the phase of the light and that of the electronic motion is not so simple as was assumed. I owe great thanks to Professor Bohr for sharing with me at an early stage the results of these more recent investigations of his—to appear soon in a paper on the conceptual structure of quantum theory—and for discussing them with me.

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