The Statistical Interpretation of Quantum Mechanics

L. E. BALLENTINE

Department of Physics, Simon Fraser University, Burnaby, B.C., Canada

The Statistical Interpretation of quantum theory is formulated for the purpose of providing a sound interpretation using a minimum of assumptions. Several arguments are advanced in favor of considering the quantum state description to apply only to an ensemble of similarily prepared systems, rather than supposing, as is often done, that it exhaustively represents an individual physical system. Most of the problems associated with the quantum theory of measurement are artifacts of the attempt to maintain the latter interpretation. The introduction of hidden variables to determine the outcome of individual events is fully compatible with the statistical predictions of quantum theory. However, a theorem due to Bell seems to require that any such hidden-variable theory which reproduces all of quantum mechanics exactly (i.e., not merely in some limiting case) must possess a rather pathological character with respect to correlated, but spacially separated, systems.

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INTRODUCTION

1.0 Preface and Outline

This article is not a historical review of how the quantum theory and its statistical interpretation came to be. That task has been admirably carried out by Max Jammer (1966) in his book *The Conceptual Development of Quantum Mechanics*, and also by van der Waerden (1967). Our point of departure can conveniently be introduced by considering the following statement made by Peierls (1967) in a review of Jammer's book:

"Chapter 7 ··· is headed 'The Copenhagen Interpretation.' ··· the phrase suggests that this is only one of several conceivable interpretations of the same theory, whereas most physicists are today convinced that the uncertainty relations and the ideas of complementarity are essential parts of the structure of quantum mechanics ····. A discussion of alternative inter-

pretive attempts is promised for a later volume. ··· [But] even the cautious word 'attempts' may be too positive a description for what are only programs ··· that are not yet clear even in outline."

If, as appears to be the case, the latter remarks by Peierls refer to the models known as hidden-variable theories (see Sec. 6), we agree that these should be treated as new theories, and that they are not new interpretations of quantum mechanics "any more than quantum mechanics is a new interpretation of classical physics." However we shall show, contrary to the view expressed by Peierls, that the Copenhagen interpretation contains assumptions which are not "essential parts of the structure of quantum mechanics," and that one such assumption is at the root of most of the controversy surrounding "the interpretation of quantum mechanics." It is the assumption that the quantum state description is the most complete possible description of an individual physical system.

An interpretation which is more nearly minimal in the sense of including all verifiable predictions of quantum theory, but without the contestable features of the Copenhagen interpretation, we shall call the *Statistical Interpretation*. The distinction between these interpretations (which share many features in common) will be made in the following sections. Suffice it to say, for now, that if we identify the Copenhagen Interpretation with the opinions of Bohr, then the Statistical Interpretation is rather like those of Einstein. Contrary to what seems to be a widespread misunderstanding, Einstein's interpretation corresponds very closely with the one which is almost universally used by physicists in practice; the additional assumptions of the Copenhagen interpretation playing no real role in the applications of quantum theory.

The outline of this paper is as follows. First we give a brief summary of the mathematical formalism of quantum theory in order to distinguish the formalism, which we accept, from the physical interpretation, which we shall examine critically. We then consider

different interpretations, and expound the Statistical Interpretation in detail.

The Secs. 2, 3, and 4 could be prefaced by Appendix *xi of Popper (1959) on the proper use of imaginary experiments, in which he points out that Gedanken experiments can be used to *criticize* a theory but not to *justify* or *prove* a theory. Our discussions of the Gedanken experiment of Einstein, Podolsky, and Rosen, of the uncertainty principle, and of the measurement process are undertaken to criticize the assumption that a state vector provides a complete description of an individual system. Although arguments of this type can refute the hypothesis being criticized, they cannot, of course, "prove" the Statistical Interpretation but can only illustrate its advantages.

Sections 5 and 6 deal with two concepts (joint probability distributions for position and momentum, and hidden variables) which have often been thought to be incompatible with quantum theory. That belief, however, was based in an essential way upon the above hypothesis which we criticize and reject. It turns out that, within certain limits, the formalism of quantum theory can be extended (*not* modified) to include these concepts within the Statistical Interpretation.

Finally we summarize our conclusions.

1.1 Mathematical Formalism of Quantum Theory

Quantum theory, and indeed any theory, can be divided [see Prugovecki (1967), or Tisza (1963)] into:

(a) A mathematical formalism consisting of a set of primitive concepts, relations between these concepts (either postulated or obtainable by given rules of deduction), and a dynamical law.

(b) Correspondence rules which relate the theoretical concepts of (a) to the world of experience.

This division is not absolute—clearly one must have a formalism in order to make correspondence rules, but unless one has at least some partial idea of correspondence rules, one would not know what one was talking about while constructing the formalism— nevertheless it is convenient for the present task.

The mathematical formalism of quantum theory is well known and can be abstracted from any of several textbooks (Dirac, 1958; Messiah, 1964). The primitive concepts are those of *state* and of *observable*.

F1 An *observable* is represented by a self-adjoint operator on a Hilbert space. It has a spectral representation,

$$R = \sum_n r_n P_n, \qquad (1.1)$$

where the P_n are orthogonal projection operators related to the orthonormal eigenvectors of R by

$$P_n = \sum_a |a, r_n\rangle\langle a, r_n|. \qquad (1.2)$$

Here the numbers r_n are the eigenvalues of R, and the parameter a labels the degenerate eigenvectors which belong to the same eigenvalue of R. The sums become integrals in the case of continuous spectra. Equation (1.1) is equivalent to the statement that an observable must possess a complete orthogonal set of eigenvectors.

F2 A *state* is represented by a state operator (also called a statistical operator or density matrix) which must be self-adjoint, nonnegative definite, and of unit trace. This implies that any state operator may be diagonalized in terms of its eigenvalues and eigenvectors,

$$\rho = \sum_n \rho_n |\phi_n\rangle\langle\phi_n|, \qquad (1.3a)$$

with

$$0 \leq \rho_n \leq 1 \qquad (1.3b)$$

and

$$\sum_n \rho_n = 1. \qquad (1.3c)$$

This state operator formalism is reviewed by Fano (1957).

F3 A *pure state* can be defined by the condition $\rho^2 = \rho$. It follows that for a pure state there is exactly one nonzero eigenvalue of ρ, say,

$$\rho_n = 1, \qquad \rho_{n'} = 0 \quad \text{for} \quad n \neq n'. \qquad (1.4)$$

In this case we have

$$\rho = |\phi_n\rangle\langle\phi_n|, \qquad (1.5)$$

and so a pure state may be represented by a vector in the Hilbert space. A general state which is not pure is commonly called a *mixed state*.

F4 The average value of an observable R in the state ρ is given by

$$\langle R \rangle = \text{Tr}\,(\rho R), \qquad (1.6)$$

where Tr means the trace of the operator in parentheses. For a pure state represented by the normalized vector $|\psi\rangle$, (1.6) reduces to $\langle R \rangle = \langle \psi | R | \psi \rangle$. By introducing the characteristic function $\langle e^{i\xi \cdot R} \rangle$, we can obtain the entire statistical distribution of the observable R in the state ρ. It follows that:

F5 The only values which an observable may take on are its eigenvalues, and the probabilities of each of the eigenvalues can be calculated. In the case of a pure state represented by the normalized vector $|\psi\rangle$, the probability of eigenvalue r_n of R is $\sum |\langle \psi | a, r_n \rangle|^2$. This is a generalization of Born's (1926) famous postulate that the square of a wave function represents a probability density.

So far we have only given necessary, but not sufficient, conditions for the mathematical representations of observables and of states. To complete the specification, the following is usually postulated:

F6 The Hilbert space is a direct sum of coherent

subspaces, within each of which (almost) every vector may represent a pure state. This is a formal statement of the *superposition principle* with allowance being made for superselection rules.[1] The set of all mixed states can be constructed from the set of all pure states using (1.3).

F7   Any self-adjoint operator which commutes with the generators of superselection rules, or equivalently, all of whose eigenvectors lie within coherent subspaces of F6, represents an observable. This postulate may be criticized on the grounds that it is difficult to imagine a procedure for observing a quantity like $(x^2 p_x{}^3 + p_x{}^3 x^2)$, which should be an observable according to this postulate. The general form of F7 is unnecessary in most, if not all practical applications, but it is used in Von Neumann's theorem (Sec. 6.1).

F8   The dynamical law or equation of motion depends in detail upon the physical system under consideration (i.e., number of degrees of freedom, whether relativistic or nonrelativistic), but in every case it can be written in the form,

$$\rho(t) = U\rho(t_0) U^{-1} \qquad (1.7a)$$

in general, or

$$|\psi(t)\rangle = U |\psi(t_0)\rangle \qquad (1.7b)$$

for a pure state, where $U = U(t, t_0)$ is a unitary operator.

The above is not intended to be an axiomatization of quantum theory, but merely a compact summary of the mathematical formalism of the theory as it exists at present and in practice. Except for the reservation noted in F7 it should be noncontroversial. Such is not the case with the correspondence rules.

1.2 Correspondence Rules

The correspondence rules must relate the primitive concepts of *state* and *observable* to empirical reality. In so doing they will provide a more specific interpretation for the averages and probabilities introduced in F4 and F5.

The natural requirement placed upon an *observable* is that we should be able to observe it. More precisely, an *observable* is a dynamical variable whose value can, in principle, be measured. For canonically conjugate variables the corresponding operators are obtained through Dirac's canonical commutation relation,

$$qp - pq = \hbar i. \qquad (1.8)$$

There is no general rule for constructing a unique operator to represent an arbitrary function $f(q, p)$ because of the noncommutability of q and p (Shewell, 1959). Fortunately the most general case does not seem

to arise in practice. We can, for example, unambiguously determine the Hamiltonian operator for any finite number of particles interacting through velocity-independent potentials in the presence of an arbitrary external electromagnetic field.

The different interpretations of quantum theory are most sharply distinguished by their interpretations of the concept of *state*. Although there are many shades of interpretation (Bunge, 1956), we wish to distinguish only two:

(I) The *Statistical Interpretation*, according to which a *pure state* (and hence also a general state) provides a description of certain statistical properties of an *ensemble* of similarily prepared systems, but need not provide a complete description of an individual system.

This interpretation is upheld, for example, by Einstein (1949), by Popper (1967), and by Blokhintsev (1968). Throughout this paper the capitalized name "Statistical Interpretation" refers to this specific interpretation (described in detail in Sec. 1.3).

(II) Interpretations which assert that *a pure state provides a complete and exhaustive description of an individual system* (e.g., an electron).

This class contains a great variety of members, from Schrödinger's original attempt to identify the electron with a wave packet solution of his equation to the several versions of the Copenhagen Interpretation.[2] Indeed many physicists implicitly make assumption II without apparently being aware that it is an additional assumption with peculiar consequences. It is a major aim of this paper to point out that the hypothesis II is unnecessary for quantum theory, and moreover that it leads to serious difficulties.

1.3 The Statistical Interpretation

The term, Statistical Interpretation, will be used throughout this paper in the specific sense here expounded, and should not be confused with the less specific usage of this term by other writers (e.g., Messiah, 1964, Chap. 1V) who do not distinguish it from the Copenhagen interpretation.

Of primary importance is the assertion that a quantum

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[1] A superselection rule (Wick *et al.*, 1952; Galindo *et al.*, 1962) is a restriction on the superposition principle. For example, a vector which is a linear combination of integer and half-integer angular momentum eigenvectors cannot represent a physical state. A *coherent* space is one in which the superposition principle has unrestricted validity.

[2] Bohr (1935, 1949) argued against Einstein, who rejected II. Heisenberg's position is somewhat unclear. One reference (1930, p. 33) contains a statement fully in accord with the Statistical Interpretation; however, in a later writing specifically in defense of the Copenhagen interpretation (1955, p. 26) he states "an individual atomic system can be represented by a wave function . . . ". His interpretation of probability (1958, Chap. 3) was based on the Aristotelian notion of "potentia", which is quite different from the Statistical Interpretation. Messiah (1964) is quite explicit (p. 152, 158) in favoring II over I. Although both claim orthodoxy, there now seems to be a difference of opinion between what may be called the Copenhagen school represented by Rosenfeld, and the Princeton school represented by Wigner (see Rosenfeld, 1968 and references therein). But since both factions appear to accept hypothesis II, our criticism will apply to both.

state (pure or otherwise) represents an *ensemble* of similarly prepared systems. For example, the system may be a single electron. Then the ensemble will be the conceptual (infinite) set of all single electrons which have been subjected to some state preparation technique (to be specified for each state), generally by interaction with a suitable apparatus. Thus a momentum eigenstate (plane wave in configuration space) represents the ensemble whose members are single electrons each having the same momentum, but distributed uniformly over all positions. A more realistic example which occurs in scattering problems is a finite wave train with an approximately well-defined wavelength. It represents the ensemble of single electrons which result from the following schematically described procedure—acceleration in a machine, the output from which can take place in only some finite time interval (due to a "chopper"), and collimation which rejects any particle whose momentum is outside certain limits. We see that a quantum *state* is a mathematical representation of the result of a certain state preparation procedure. Physical systems which have been subjected to the same state preparation will be similar in some of their properties, but not in all of them (similar in momentum but not position in the first example). Indeed the physical implication of the uncertainty principle (discussed in detail in Sec. 3) is that no state preparation procedure is possible which would yield an ensemble of systems identical in all of their observable properties. Thus it is most natural to assert that a quantum state represents an ensemble of similarily prepared systems, but does not provide a complete description of an individual system.

When the physical system is a single particle, as in the above examples, one must not confuse the *ensemble*, which is a conceptual set of replicas of one particle in its experimental surroundings, with a *beam* of particles, which is another kind of (many-particle) system. A beam may simulate an ensemble of single-particle systems if the intensity of the beam is so low that only one particle is present at a time.

The ensembles contemplated here are different in principle from those used in statistical thermodynamics, where we employ a representative ensemble for calculations, but the result of a calculation may be compared with a measurement on a single system. Also there is some arbitrariness in the choice of a representative ensemble (microcanonical, canonical, grand canonical). But, in general, quantum theory predicts nothing which is relevant to a single measurement (excluding strict conservation laws like those of charge, energy, or momentum), and the result of a calculation pertains directly to an ensemble of similar measurements. For example, a single scattering experiment consists in shooting a single particle at a target and measuring its angle of scatter. Quantum theory does not deal with such an experiment, but rather with the statistical distribution (the differential cross section) of the results of an ensemble of similar experiments. Because this ensemble is not merely a representative or calculational device, but rather it can and must be realized experimentally, it does not inject into quantum theory the same conceptual problems posed in statistical thermodynamics.

In general quantum theory will not predict the result of a measurement of some observable R. But the probability of each possible result r_n, calculated according to F5, may be verified by repeating the state preparation and the measurement many times, and then constructing the statistical distribution of the results. As pointed out by Popper (see Körner, 1957, p. 65, p. 88), one should distinguish between the *probability*, which is the relative frequency (or measure) of the various eigenvalues of the observable in the conceptual infinite ensemble of all possible outcomes of identical experiments (the sample space), and the *statistical frequency* of results in an actual sequence of experiments. The probabilities are properties of the state preparation method and are logically independent of the subsequent measurement, although the statistical frequencies of a long sequence of similar measurements (each preceded by state preparation) may be expected to approximate the probability distribution. If hypothesis I is adopted, we may say simply that the probabilities are properties of the *state*.

The various interpretations of a quantum *state* are related to differences in the interpretation of probability (see Popper, 1967 for a good exposition of this point). In contrast to the Statistical Interpretation, some mathematicians and physicists regard probability as a measure of knowledge, and assert that the use of probability is necessitated only by the incompleteness of one's knowledge. This interpretation can legitimately be applied to games like bridge or poker, where one's best strategy will indeed be influenced by any knowledge (accidentally or illegally obtained) about an opponent's cards. But physics is not such a game, and as Popper has emphasized, one cannot logically deduce new and verifiable knowledge—statistical knowledge—literally from a lack of knowledge.

Heisenberg (1958), Chap. 3, combined this "subjective" interpretation with the Aristotelian notion of "potentia." He considers a particle to be "potentially present" over all regions for which the wave function $\psi(\mathbf{r})$ is nonzero, in some "intermediate kind of reality," until an act of observation induces a "transition from the possible to the actual." In contrast, the Statistical Interpretation considers a particle to always be at some position in space, each position being realized with relative frequency $|\psi(\mathbf{r})|^2$ in an ensemble of similarily prepared experiments. The "subjective" and Aristotelian ideas are primarily responsible for the suggestion that the observer plays a peculiar and essential role in quantum theory. Whether or not they can be consistently developed, the existence of the Statistical Interpretation demonstrates that they are *not necessary*,

and in my opinion they bring with them no advantages to compensate for the additional metaphysical complication.

If the expression "wave–particle duality" is to be used at all, it must not be interpreted literally. In the above-mentioned scattering experiment, the scattered portion of the wave function may be equally distributed in all directions (as for an isotropic scatterer), but any one particle will not spread itself isotropically; rather it will be scattered in some particular direction. Clearly the wave function describes not a single scattered particle but an ensemble of similarily accelerated and scattered particles. At this point the reader may wonder whether a statistical particle theory can account for interference or diffraction phenomena. But there is no difficulty. As in any scattering experiment, quantum theory predicts the statistical frequencies of the various angles through which a particle may be scattered. For a crystal or diffraction grating there is only a discrete set of possible scattering angles because momentum transfer to and from a periodic object is quantized by a multiple of $\Delta p = h/d$, where Δp is the component of momentum transfer parallel to the direction of the periodic displacement d. This result, which is obvious from a solution of the problem in momentum representation, was first discovered by Duane (1923), although this early paper had been much neglected until its revival by Landé (1955, 1965). There is no need to assume that an electron spreads itself, wavelike, over a large region of space in order to explain diffraction scattering. Rather it is the crystal which is spread out, and the electron interacts with the crystal as a whole through the laws of quantum mechanics. For a longer discussion of this and related problems such as the two-slit experiment, see Landé (1965).[3] In every case a diffraction pattern consists of a statistical distribution of discrete particle events which are separately observable if one looks in fine enough detail. In the words of Mott (1964, p. 409), "Students should not be taught to doubt that electrons, protons and the like are particles ··· The wave cannot be observed in any way than by observing particles."

Although we shall discuss his work in greater detail below, we should emphasize here the great contribution of Einstein to this subject. His "Reply to Criticisms" (Einstein, 1949), expressed very clearly his reasons for accepting a purely statistical (ensemble) interpretation of quantum theory, and rejecting the assumption that the state vector provided an exhaustive description of the individual physical system.

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[3] I note in passing that Landé's ambitious program to derive all of quantum theory from a few simple principles has not yet been completely successful. Reviewers of his book (Shimony, 1966; Witten, 1966) have pointed out additional assumptions implicit in his argument which are virtually equivalent to assuming some of the results he wishes to derive. However, these criticisms do not detract from his discussion of diffraction scattering.

2. THE THEOREM OF EINSTEIN, PODOLSKY, AND ROSEN

The tenability of hypothesis II, Sec. 1.2, was challenged in a paper by Einstein, Podolsky, and Rosen (1935) (abbreviated EPR) entitled "Can Quantum-Mechanical Description of Reality Be Considered Complete?" Their argument is often referred to as the Paradox of EPR, as though it ought to be capable of resolution as, say, Zeno's paradox. [Rosenfeld (1968) has scurrilously referred to it as the EPR fallacy.] We shall show however that, properly interpreted, it is a well defined theorem, paradoxical only to the extent that the reader may not have expected the conclusion.

In order to precisely answer the question posed in their title, EPR introduce the following definitions:

D1   A *necessary* condition for a *complete* theory is that *"every element of physical reality must have a counterpart in the physical theory."*

D2   A *sufficient* condition for identifying an *element of reality* is, *"If, without in any way disturbing a system, we can predict with certainty* (i.e., *with probability equal to unity)*[4] *the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."*

Their argument then proceeds by showing, through consideration of a thought experiment, that two noncommuting observables should, under suitable conditions, both be considered elements of reality. Since no state vector can provide the value of both of these observables, they conclude that the quantum state vector cannot completely describe an individual system, but only an ensemble of similarily prepared systems.

2.1 A Thought Experiment and the Theorem

The experiment described below was introduced by Bohm (1951, p. 614ff). By considering measurements of spins rather than of particle positions and momenta (as in the original EPR experiment), we avoid any unnecessary complications with the position–momentum uncertainty principle which may arise from the minimum degree of particle trajectory definition needed to perform an experiment.

Two particles of spin one-half are prepared in an unstable initial state of total spin zero. The pair separates, conserving total spin, and one of the particles (which are chosen to be distinguishable, for convenience) passes through the inhomogeneous magnetic field of a Stern–Gerlach apparatus (see Fig. 1).

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[4] As a nearly pedantic refinement, we would prefer to replace this phrase by *with probability* $1-\epsilon$, *where* ϵ *may be made arbitrarily small*. This allows for arbitrarily accurate approximations to problems which may not be formally solvable, and it allows one to avoid certain irrelevant criticisms. (See Footnote 5).

The uncertainty principle is trivially satisfied since we require only that $p_x = p_z = 0$ approximately, and that Δy be small enough so we know which particle has entered the magnetic field.

The translational motion of the particles can be treated classically, and the spin can be described by a spin Hamiltonian in which the magnetic field at the position of the particle is treated as a function of time,

$$H = V_{12} + V_H. \qquad (2.1)$$

Here $V_{12} = V(\mathbf{r}_{12}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ is the interaction between the two particles, and V_H is the interaction with the magnetic field.

$$V_H(t) = zH'\sigma_{1z}, \quad \text{for} \quad 0 < t < \tau,$$
$$= 0, \qquad \text{for} \quad t < 0 \text{ or } t > \tau, \qquad (2.2)$$

where τ is the transit time of the particle through the magnetic field which is of gradient H' and is directed along the z axis.

Omitting the position-dependent factors, we can write the initial state vector, a spin singlet, as

$$\psi(0) = \{u_+(1)u_-(2) - u_-(1)u_+(2)\}/\sqrt{2}. \qquad (2.3)$$

Here u_+ and u_- are the two-component spinors corresponding to the eigenvalues ± 1 of σ_z, and the arguments 1 or 2 refer to the two particles. Since

$$\psi(t) = \exp\left\{\left(-\frac{i}{\hbar}\right)\int_0^t H(t')\,dt'\right\}\psi(0), \qquad (2.4)$$

the effect of the interparticle interaction V_{12} will be negligible if

$$|V_{12}|\,\tau/\hbar \ll 1, \qquad (2.5)$$

a condition which can be realized by making the transit time τ short enough.[5] By $|V_{12}|$ we mean the modulus of the relevant matrix elements of V_{12} in (2.4).
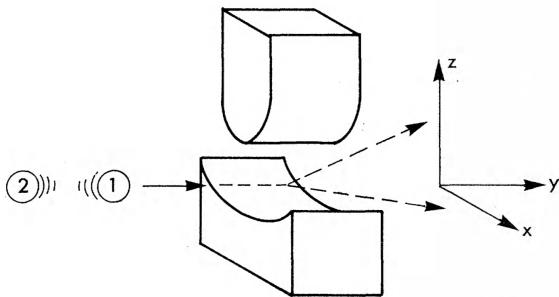

FIG. 1. Schematic illustration of the apparatus for the Einstein, Podolsky, and Rosen experiment.

[5] This overcomes the objection of Sharp (1961) that any interaction (even gravitational) would prevent us from writing $\psi(t)$ as a product of two factors; one for particle 1, and one for particle 2. Such an objection is quite irrelevant because any state vector can be expressed as a linear combination of product-type basis vectors, and the interaction only affects the time dependence of the expansion coefficients which we treat to an arbitrary degree of precision.

At time $t = \tau$, after the particle has interacted with the magnetic field, the state vector will be

$$\psi(\tau) = \{\exp\left(-iH'\tau z_1/\hbar\right)u_+(1)u_-(2)$$
$$- \exp\left(iH'\tau z_1/\hbar\right)u_-(1)u_+(2)\}/\sqrt{2}. \qquad (2.6)$$

The result of the interaction is to produce a correlation between z components of momentum and spin of particle 1, and spin of particle 2. If $p_{1z} = -H'\tau$, then $\sigma_{1z} = +1$, and $\sigma_{2z} = -1$; or if $p_{1z} = H'\tau$ then $\sigma_{1z} = -1$ and $\sigma_{2z} = +1$. It is only necessary to make the magnetic field gradient H' large enough so that the two values of p_{1z} are unambiguously separated.

We have thus shown that the z component of the spin of particle 2 can be determined to an arbitrarily high degree of accuracy by a measurement which, because of the spacial separation and negligible effect of V_{12}, does not in any way disturb particle 2. Thus according to definition D2, σ_{2z} is an *element of reality*.

However the initial singlet state is invariant under rotation, and can equally be expressed as

$$\psi(0) = \{v_+(1)v_-(2) - v_-(1)v_+(2)\}/\sqrt{2}, \qquad (2.7)$$

where the spinors v_+ and v_- are eigenvectors of σ_x. If the Stern–Gerlach magnet were rotated so that the field was directed along the x axis, then by an identical argument we would conclude that σ_{2x} was an *element of reality*. Since no state vector can provide a value for both of the noncommuting observables σ_{2x} and σ_{2z}, EPR conclude, in accordance with D1, that the state vector does not provide a complete description of an individual system.

One might try to avoid this conclusion by adopting an extreme positivist philosophy, denying the *reality* of both σ_{1z} and σ_{2z} until the measurement has actually been performed. But this entails the unreasonable, essentially solipsist position that the reality of particle 2 depends upon some measurement which is not connected to it by any physical interaction.

In any case, the conclusion can be stated, as was first done by Einstein (1949, p. 682), as the following *theorem*:

The following two statements are incompatable:

(1) The state vector provides a complete and exhaustive description of an individual system;

(2) The real physical conditions of spatially separated (noninteracting) objects are independent.

Of course one is logically free to accept either one of these statements (or neither). Einstein clearly accepted the second, while Bohr apparently favored the first. The importance of the EPR argument is that it proved for the first time that assuming the first statement above *demands* rejection of the second, and vice versa, a fact that was not at all obvious before 1935, and which may not be universally realized today.

2.2 Discussion of the EPR Theorem

It should be emphasized that the result of the EPR experiment is in no way paradoxical. The implication of (2.6) in the Statistical Interpretation is that if the experiment is repeated many times we should obtain the result $p_{1z} = -H'\tau$, $\sigma_{1z} = +1$, $\sigma_{2z} = -1$ in about one-half of the cases, and the opposite result in the other half of the cases. This correlation between the spins of the two particles will be the same no matter which components are measured.

Another essential point, which has not always been realized, is that the EPR theorem in no way contradicts the mathematical formalism of quantum theory. It was only intended to exhibit the difficulties which follow from the assumption that the state vector completely describes an individual system; an interpretive assumption which is not an integral part of the theory. In his famous reply to EPR, Bohr (1935) said, "Such an argumentation [as that of EPR], however, would hardly seem suited to affect the soundness of quantum-mechanical description, which is based on a coherent mathematical formalism ····." It would appear that Bohr failed to distinguish between the mathematical formalism of quantum theory, and the Copenhagen interpretation of that theory. Einstein's criticism was directed only at the latter, not at the former. Conversely the self-consistency and empirical success of the former are no defense against specific criticism of the latter. Such an erroneous perception of the EPR paper as being an attack on quantum theory itself may explain (psychologically) the often repeated statement that Bohr had successfully refuted their argument. In fact Bohr's paper offers no real challenge to the validity of the EPR theorem as stated above, nor does the EPR theorem pose any "paradox" or threat to the validity of quantum theory.

Bohr's reply to EPR is really a criticism of their definitions of *completeness* and *physical reality*, but it is of a rather imprecise character. A satisfactory pursuit of this line of attack should first admit the validity of the EPR theorem with their definitions. Second it should propose alternative definitions with arguments in favor of the superiority of the new definitions. Finally it should show that the state vector provides a *complete* description of *physical reality* in terms of the new definitions. Bohr has done none of these.

In the succeeding years numerous comments on the EPR paper have been published, many of which are less than satisfactory. A discussion of a paper by Furry (1936) is given in Footnote 13, Sec. 4.3 of this paper. Breitenberger (1965) has criticized many authors for contributing to confusion, including Bohr for suggesting that the observation of particle 1 "creates" the physically real state of particle 2—a position approaching the absurdity of solipsism.[6] Breitenberger correctly

emphasizes that there is nothing paradoxical about the EPR experiment, which is a prototype for many coincidence experiments, but he seems to forget the original purpose of the EPR argument.

Several people have proposed experiments similar in principle to that of EPR (Day, 1961; Inglis, 1961; Bohm and Aharonov, 1957). While these experiments are interesting in their own right, it should be emphasized that their results will not distinguish between Einstein's and Bohr's interpretations of quantum theory. If the results of experiments were to differ from the theoretical predictions, this would contradict the formalism of quantum theory itself, not just one of the interpretations.

3. THE UNCERTAINTY PRINCIPLE

3.1 Derivation

To a given *state* there will, in general, correspond a statistical distribution of values for each *observable*. A suitable measure of the width of the distribution for an *observable* A is the variance,

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle, \qquad (3.1)$$

where ΔA is known as the standard deviation of the distribution. Although *states* exist (at least as mathematical idealizations) for which the variance of the distribution for any one observable is arbitrarily small, it can easily be shown, as was first done by Robertson (1929) and Schrödinger (1930), that the product of the variances of the distributions of two observables A and B has a lower bound,

$$(\Delta A)^2 (\Delta B)^2 \geq (\tfrac{1}{2}\langle AB + BA \rangle - \langle A \rangle \langle B \rangle)^2 + \tfrac{1}{4}\langle C \rangle^2, \quad (3.2)$$

where $AB - BA = iC$. The first term may vanish, but for canonically conjugate observables [see Eq. (1.8)] we must always have

$$\Delta q \Delta p \geq \hbar/2. \qquad (3.3)$$

The meaning of these results is unambiguous. The averages of quantum theory (postulate F4) are realized by performing the same measurement on many similarly prepared systems (or equivalently by performing the measurement many times on the same system which must be resubmitted to the same state preparation before each measurement). In order to measure ΔA (or ΔB) one must measure A (or B) on many similarily prepared systems, construct the statistical distribution of the results, and determine its standard deviation. The results (3.2) and (3.3) assert that for any particular *state* (i.e., state preparation) the product of the widths of the distributions of A measurements and of B measurements may not be less than some lower limit. A term such as *the statistical dispersion principle* would really be more appropriate for these results than the traditional name, *uncertainty principle*. A discussion similar to this is given by Margenau (1963).

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[6] I have not found an explicit statement to this effect in Bohr's writings, although essentially this position has been taken by some followers of Bohr.

3.2 Relation to Experiments

There exist many widespread statements of the uncertainty principle which are different from the one that is derived from statistical quantum theory. Very common is the statement that *one cannot measure the two quantities q and p simultaneously without errors whose product is at least as large as ℏ/2*. This statement is often supported by one or both of the following arguments:

(i) A measurement of q causes an unpredictable and uncontrollable disturbance of p, and vice versa. [This was first proposed by Heisenberg (1927) and is widely repeated in text books].

(ii) The position and momentum of a particle do not even exist with simultaneously and perfectly well defined (though perhaps unknown) values (Bohm, 1951, p. 100).

Clearly the statistical dispersion principle and the common statement of the uncertainty principle are not equivalent or even closely related. The latter refers to *errors* of *simultaneous* measurements of q and p on *one* system, and it is plausible that one of these measurements could cause an error in the other. On the other hand, the former refers to *statistical spreads* in *ensembles* of measurements on similarily prepared systems. But only one quantity (either q or p) is measured on any one system, so there is no question of one measurement interfering with the other. Furthermore the standard deviations Δq and Δp of the statistical distributions cannot be determined unless the errors of the individual measurements, δq and δp are much smaller than the standard deviations (see Fig. 2). These points have been emphasized by Margenau (1963) and by Popper (1967). Prugovecki (1967) has pointed out that, far from restricting simultaneous measurements of non-commuting observables, quantum theory does not deal with them at all; its formalism being capable only of statistically predicting the results of measurements of one observable (or a commutative set of observables). In order to deal with simultaneous measurements he proposes an extension of the mathematical formalism, to which we shall return in Sec. 5 of this paper.

We now consider to what extent the common statement of the uncertainty principle may be true, even
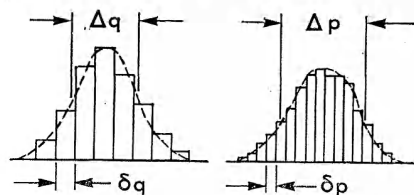
FIG. 2. Illustration of the uncertainty principle. The two histograms represent the frequency distributions of independent measurements of q and p on similarly prepared systems. There are, in principle, no restrictions on the precisions of individual measurements δq and δp, but the standard deviations will always satisfy $\Delta q\Delta p\geq\hbar/2$.
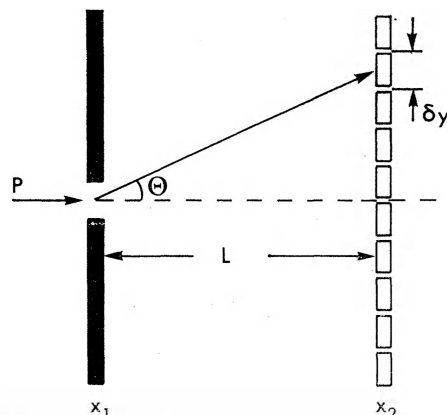
FIG. 3. An experimental arrangement designed to simultaneously measure y and p_y, such that the error product $\delta y\delta p_y$ can be arbitrarily small.

though it is not derivable from quantum theory. Argument (ii) is easily seen to be unjustified. It is based on the obvious fact that a wave function with a well defined wavelength must have a large spacial extension, and conversely a wave function which is localized in a small region of space must be a Fourier synthesis of components with a wide range of wavelengths. Using de Broglie's relation between momentum and wavelength, $p=h/\lambda$, it is then asserted that a particle cannot have definite values of both position and momentum at any instant. But this conclusion rests on the almost literal identification of the particle with the wave packet (or what amounts to the same thing, the assumption that the wave function provides an exhaustive description of the properties of the particle). The untenable nature of such an identification is shown by the example of a particle incident upon a semitransparent mirror with detectors on either side. The particle will either be reflected or transmitted without loss of energy, whereas the wave packet is divided, half its amplitude being transmitted and half reflected. A consistent application of the Statistical Interpretation yields the correct conclusion that the division of the wavepacket yields the relative probabilities for transmission and reflection of particles. But there is no justification for assertion (ii).

Argument (i) must be considered more seriously since Heisenberg (1930) and Bohr (1949) have given several examples for which it is apparently true. However, Fig. 3 shows a simple experiment for which it is not valid. A particle with known initial momentum p passes through a narrow slit in a rigid massive screen. After passing through the hole, the momentum of the particle will be changed due to diffraction effects, but its energy will remain unchanged. When the particle strikes one of the distant detectors, its y coordinate is thereby measured with an error δy. Simultaneously this same event serves to measure the y component of momentum, $p_y=p\sin\theta$, with an error δp_y which may be made arbitrarily small by making the distance L

arbitrarily large. Clearly the product of the errors δp_y need not have any lower bound, and so the common statement of the uncertainty principle given above cannot be literally true. One may raise the objection that p_y has not been measured, but only defined in the above equation. However this method of measuring momentum by means of geometrical inference from a position measurement is universally employed in scattering experiments. It rests upon the assumption of linear motion in a field-free region (Newton's First Law), which remains valid in quantum mechanics (at least for L much greater than a de Broglie wavelength).

The distinction between experiments which satisfy and which violate the uncertainty principle can be clarified with the help of the concepts of *state preparation* and *measurement*; the distinction between these having been emphasized by Margenau (1958, 1963) and by Progovecki (1967).

State preparation refers to any procedure which will yield a statistically reproducible ensemble of systems. The concept of *state* in quantum theory (see Sec. 1.3) can be considered operationally as an abbreviation for a description of the state preparation procedure.[7] Of course there may be more than one experimental procedure which yields the same statistical ensemble, i.e., the same state. An important special case (which is sometimes incorrectly identified with measurement) is a filtering operation, which ensures that if a system passes through the filter it must immediately afterward have a value of some particular observable within a restricted range of its eigenvalue spectrum.

On the other hand, *measurement* of some quantity R for an individual system means an interaction between the system and a suitable apparatus, so that we may infer the value of R (within some finite limits of accuracy) which the system had immediately before the interaction (or the value of R which the system would have had if it had not interacted, allowing for the possibility that the interaction will disturb the system).

The essential distinctions between the two concepts are that state preparation refers to the future, whereas measurement refers to the past; and equally important, that measurement involves *detection* of a particular system, whereas state preparation provides conditional information about a system *if* it passes through the apparatus.

The *statistical dispersion principle* (3.2, 3.3), which follows from the formalism of quantum theory, is a statement about the minimum dispersion possible in any state preparation. It is significant that experiments which satisfy the uncertainty principle can be employed as state preparations, whereas experiments which violate the uncertainty principle cannot.

Consider, for example, Heisenberg's (1930, p. 21)

famous microscope for measuring the position of an electron. If the angular aperture of the microscope is ϵ and the wavelength of light used is λ, then the accuracy of the position measurement will be limited to

$$\delta x = \lambda/\sin \epsilon, \qquad (3.4)$$

by the resolving power of the instrument. (We use the symbol δ to indicate the uncertainty or error in an individual measurement, while Δ refers to the standard deviation of an ensemble of similar measurements). Because the direction of the scattered photon is unknown within a cone of angle ϵ, the x component of momentum of the electron will be changed by some unknown increment in the range $\pm \sin \epsilon (h/\lambda)$. Note that this experiment is *not* an example of simultaneous measurement of x and p_x. Only x is measured here.

Suppose now that we have a state preparation apparatus which will produce an ensemble of electrons with a negligible spread in momentum (i.e., $\Delta p_x \approx 0$). Suppose that we try to select from this ensemble a subensemble which has as small as possible a spread Δx. We do this by measuring the x coordinate of each particle and selecting those which fall within the minimum resolvable range δx of a particular value, say, x'. Thus our subensemble will have a spread

$$\Delta x = \lambda/\sin \epsilon. \qquad (3.5)$$

Because the recoil momentum absorbed by an electron may vary, the spread in momentum among the members of this subensemble will be

$$\Delta p_x \approx \sin \epsilon (h/\lambda). \qquad (3.6)$$

Hence the ensemble which we have selected will have statistical dispersion of magnitude $\Delta x \Delta p_x \approx h$.

Contrast the above result with the experiment illustrated in Fig. 3. In this case we are able to determine the values of the position and momentum of a particle to an accuracy $\delta y \delta p_y \ll \hbar$. But this information refers to the motion of the particle during a certain interval *before* the measurement. We cannot use it to generate an ensemble of particles with statistical spreads $\Delta y \Delta p_y \ll \hbar$ because the scattering of the particles by the detector will cause a spread of momentum in the final ensemble, $\Delta p_y \gg \delta p_y$. If we try to avoid this scatter by removing one of the detectors from the array at the plane $x = x_2$, and inferring from a negative response of all the remaining detectors that the particle passed through the hole, then the momentum spread of the ensemble formed in this way will not be δp_y (the error of measurement described above). Rather it will be $\Delta p_y \approx \delta y/\hbar$ (due to diffraction effects). The spread of particle positions in the ensemble will be $\Delta y \approx \delta y$ (the size of the hole), and so the *statistical dispersion principle* will be satisfied.

In conclusion, the uncertainty principle restricts the degree of statistical homogeneity which it is possible to achieve in an ensemble of similarly prepared systems,

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[7] Lamb (1969) has described an idealized method of preparing an arbitrary single-particle state.

and thus it limits the precision which future predictions for any system can be made. But it does not impose any restriction on the accuracy to which an event can be reconstructed from the data of both state preparation and measurement in the time interval between these two operations. Heisenberg (1930, p. 20) made nearly this distinction in his statement, "··· the uncertainty relation does not refer to the past." His subsequent remark, "It is a matter of personal belief whether ··· the past history of the electron [as inferred from an experiment like that of Fig. 3] can be ascribed any physical reality or not,"—based on the fact that it cannot be reconfirmed by any other future measurement, seems unduely cautious. In fact, the majority of real physical measurement are of just this type. Indeed for every future oriented experimental operation which yields verifiable information, there must be another past oriented operation whose function is not to be verifiable but to verify (Popper, 1967, pp. 25–28). It is just this essential fact which is emphasized by the distinction between *state preparation* and *measurement*.

3.3 Angular and Energy–Time Relations

An apparent contradiction of the uncertainty principle, which is frequently rediscovered by bright students, concerns the polar angle φ and the z component of angular momentum L_z. If L_z is represented by $-i\hbar\partial/\partial\varphi$, then it would appear that $[\varphi, L_z]=i\hbar$, from which it would follow that

$$\Delta\varphi\Delta L_z\geq\hbar/2 \quad \text{(wrong)}. \qquad (3.7)$$

But (3.7) is obviously wrong since ΔL_z can be arbitrarily small, and φ has a meaningful range of only 2π radians; hence, $\Delta\varphi\leq2\pi$.

The resolution of the apparent paradox, as pointed out by Judge and Lewis (1963) and by Susskind and Glogower (1964), lies in the observation that the operator $-i\hbar(\partial/\partial\varphi)$ is Hermitian only on the space of functions of φ which have period 2π, and multiplication by φ destroys this periodicity property. In order to deduce a correct uncertainty relation, one must replace the coordinate φ by some periodic function. This can be done in many ways but the most convenient seems to be the use of $\cos\varphi$ and $\sin\varphi$ (Carruthers and Nieto, 1968), from which it can be shown that

$$(\Delta L_z)^2[(\Delta\cos\varphi)^2+(\Delta\sin\varphi)^2]$$
$$\geq(\hbar/2)^2(\langle\sin\varphi\rangle^2+\langle\cos\varphi\rangle^2). \quad (3.8)$$

The derivation of the energy–time uncertainty relation cannot follow the standard form (3.2) because time is not usually represented by an operator. In fact one can show (Susskind and Glogower, 1964) that, if the energy spectrum has a lower bound, then there does not exist a Hermitian operator which is canonically conjugate to the Hamiltonian in the sense of (1.8). However, the following result can be deduced from

(3.2) (Messiah, 1964, p. 319),

$$\Delta A\Delta E\geq\tfrac{1}{2}\hbar\,|\,d\langle A\rangle/dt\,|, \qquad (3.9)$$

where A is an arbitrary observable, and the averages are calculated for an arbitrary time-dependent state. If one defines a characteristic time for the system and the state as

$$\tau=\min_{(A)}\{\Delta A\,|\,d\langle A\rangle/dt\,|^{-1}\}, \qquad (3.10)$$

then one may write

$$\tau\Delta E\geq\tfrac{1}{2}\hbar. \qquad (3.11)$$

Clearly this result does not imply that one cannot measure the energy of a system exactly at an instant of time, as is sometimes stated, but rather that the spread of energies associated with any state is related to the characteristic rate of change associated with the same state.[8]

4. THE THEORY OF MEASUREMENT

The desirability of an analysis of the measurement process by means of quantum theory was perhaps first indicated through Bohr's insistence that the *"whole experimental arrangement,"* (Bohr, 1949, p. 222), object plus all apparatus, must be taken into account in order to specify well-defined conditions for an application of the theory. However he never carried this program to its logical conclusion; the description of both the object and the measuring apparatus by the formalism of quantum theory. Such an analysis is useful for several reasons.

It has frequently been asserted that the acts of measurement or observation play a different role in quantum theory than in the rest of physics. Such claims should be critically analyzed.

Because the measurement apparatus usually (always?) contains a final stage to which classical mechanics is applicable, the measurement of a quantal (i.e., essentially nonclassical) object by such an apparatus involves the union of micro and macrophysics in an essential way. Any proposed revisions at the microscopic level, of the concept of physical reality or of the role of the observer will meet a critical test here. It is widely believed that Ehrenfest's theorem (Messiah, p. 216) proves the consistency of quantum theory with the classical limit, but this is only partly true. Ehrenfest's theorem demonstrates that the average values of observables obey the classical equations of motion provided the quantum state function is such that statistical deviations of the basic observables (co-ordinates and momenta) from their averages are negligible. But this leaves open the question of whether or not the state of a macroscopic object coupled to microscopic objects, as calculated from the quantal equation of motion for realistic but general initial

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[8] After completion of this manuscript, a paper by Allcock (1969) containing a fuller discussion of energy-time relations appeared.

conditions, will necessarily possess this property. As will be shown, this appears not to be so in the case of measurement.

It is not the compatibility of the classical and quantal equations of motion (in the appropriate limit) which concerns us but rather the compatibility of the classical and quantal concepts of *state*. Thus an analysis of measurement will be very helpful in deciding the question raised in Sec. 1.2, that is whether a quantum *state* describes an individual system (object plus apparatus in this case), or whether it must refer only to an ensemble of systems.

4.1 Analysis of the Measurement Process

The essence of a measurement is an interaction between the object to be measured and a suitable apparatus so that a correspondence is set up between the initial state of the object and the final state of the apparatus. This interaction may or may not change the value of the observable being measured, the final state of the object being of no significance to the success of the measurement.

Suppose we wish to measure the observable R of the object I, for which there must be a complete set of eigenvectors,

$$R \mid I; r \rangle = r \mid I; r \rangle.$$

Denote a set of states for the apparatus II by $\mid II; \alpha \rangle$, where the eigenvalue α is the appropriate "pointer position" of the apparatus, and $\mid II; 0 \rangle$ is the initial premeasurement state. The interaction between the object and the apparatus ideally should, if it is to function as a measurement, set up a unique correspondence between the initial value of r and the final value of α, α_r, for example. That is, if the initial state for the system I+II is $\mid I; r \rangle \mid II; 0 \rangle$, then the equation of motion must lead to the final state

$$U \mid I; r \rangle \mid II; 0 \rangle = \mid I; r \rangle \mid II; \alpha_r \rangle, \qquad (4.1a)$$

if the observable R is not changed by the interaction[9,10]; or in the general case

$$U \mid I; r \rangle \mid II; 0 \rangle = \mid I; \phi_r \rangle \mid II; \alpha_r \rangle, \qquad (4.1b)$$

where U is the time-development operator for the duration of the interaction between I and II. Note that the vectors $\mid I; \phi_r \rangle$ need not be orthogonal since or-

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[9] Many authors give the unwarranted impression that this highly special case is general.

[10] Araki and Yanase (1960) have shown that unless R commutes with all universal additively conserved quantities, then no interaction exists which could satisfy (4.1a). However, they also show that this equation can nevertheless be satisfied to an arbitrary degree of accuracy, provided the apparatus is made arbitrarily large. This fact is illustrated in the measurement of the z component of spin by a Stern–Gerlach apparatus (Sec. 2.1), which used an external magnetic field to break the conservation of S_x and S_y. The source of this magnetic field is rigidly fixed to the earth, and so the apparatus is effectively infinite, as it must be to satisfy the theorem of Araki and Yanase. See also Yanase (1961).

thogonality of the final states corresponding to different r values is guaranteed by the fact that $\alpha_{r'} \neq \alpha_r$, for $r' \neq r$.

Suppose now that the initial state of the object is not an eigenvector of R, but rather some linear combination

$$\mid I; \psi \rangle = \sum_r \mid I; r \rangle \langle r \mid \psi \rangle. \qquad (4.2)$$

From (4.1) and the linearity of the equation of motion, the final state for the system must be

$$U \mid I; \psi \rangle \mid II; 0 \rangle = \sum_r \langle r \mid \psi \rangle \mid I; \phi_r \rangle \mid II; \alpha_r \rangle$$

$$= \mid I + II; f \rangle, \quad \text{say.} \qquad (4.3)$$

A specific illustration of this general analysis is provided by the EPR experiment discussed in Sec. 2.1. We may consider the measured quantity R to be the spin of particle 2, and the "pointer" of the apparatus to be the momentum of particle 1 (assuming that two detectors are placed so as to determine its sign).

According to the Statistical Interpretation and F5 of the formalism, the probability of the apparatus "pointer" being α_r at the end of an experiment is $\mid \langle r \mid \psi \rangle \mid^2$. That is to say, if the experiment were repeated many times, always preceded by the same state preparation for both object and apparatus, the relative frequency of the value α_r would be $\mid \langle r \mid \psi \rangle \mid^2$. That this should be identical with the probability of the object having had the value r for the dynamical variable R immediately before interacting with the apparatus II, is just the criterion that II functions as a measuring apparatus for R.

4.2 The Difficulties of the "Orthodox" Interpretation

The discussion of the analysis of measurement according to the Statistical Interpretation was so simple and natural that further comment almost seems redundant. But if instead of the basic assumption of the Statistical Interpretation that a state vector characterizes an *ensemble* of similarily prepared systems, one assumes that a state provides a complete description of an *individual* system, then the situation is quite different. Unfortunately this assumption has been made so frequently, more often tacitly than not, that it and the resulting point of view have come to be considered "orthodox".

The "orthodox" quantum theory of measurement was formalized by J. Von Neumann (1955) and has been clearly reviewed by Wigner (1963). The first difficulty encountered is that if one tries to interpret the final-state vector (4.3) as completely describing one system (object plus apparatus), then one is led to say that the pointer of the apparatus has no definite position, since (4.3) is a coherent superposition of vectors corresponding to different values of α. But the pointer can be a quite classical object and it makes no sense to say that it has no position at any instant of time, especially since the different values of α_r are macro-

scopically distinguishable. Thus if we attribute the state vector to an individual system, we are inevitably led to classically meaningless states for a classical object.[11]

To circumvent this difficulty, the "orthodox" theory assumes that, in addition to the continuous evolution of the state vector according to the equation of motion, there is also an unpredictable discontinuous *"reduction of the state vector"* upon "measurement," from (4.3) to

$$| \text{I}; \phi_{r'} \rangle \, | \text{II}; \alpha_{r'} \rangle, \qquad (4.4)$$

where $\alpha_{r'}$ is the pointer position which is actually observed. Proponents of this point of view also usually assume that the value of the observable R is not changed by measurement, so that in place of (4.4) they would obtain

$$| \text{I}; r' \rangle \, | \text{II}; \alpha_{r'} \rangle, \qquad (4.4')$$

but this point is not essential. The word "measurement" is enclosed in quotes because its use in this peculiar context of the "orthodox" theory is not equivalent to the more physical definition of measurement given in Sec. 3.

It should be emphasized that this *reduction* cannot arise from the equation of motion. The form of (4.3), involving a superposition of macroscopically distinct pointer positions, depends only upon the linearity of the equation of motion and not upon any of our simplifying assumptions. For example, one might object that the pointer position eigenvalue α is not sufficient to label a unique state vector for the apparatus, since there are a huge number of commuting observables. But this means only that each vector $| \text{II}; \alpha \rangle$ must be replaced by a set of vectors, all the sets being mutually orthogonal. Komar (1962) has shown that even with these extra degrees of freedom, a final state of the form (4.4) is not possible.[12] If the *reduction of the state vector* is to enter the theory at all, it must be introduced as a special postulate (often called the *projection postulate*).

If one merely applied the projection postulate to the object I, one would say that upon "measurement" the state changes discontinuously from (4.2) to $| \text{I}; r' \rangle$, where r' is the "measured" value of the observable R. On the other hand, the analysis of the interaction between the object I and the apparatus II leads to the state (4.3). To avoid a direct contradiction, Von

Neumann introduced the observer III, who supposedly performs a "measurement" on I+II (this "measurement" however is merely an observation), and thus *reduces* the state vector to (4.4) or (4.4'). He then shows that it makes no difference whether the apparatus II is considered to be part of the observer or part of the object being observed.

Von Neumann's theory of measurement is very unsatisfactory. It suggests that the passive act of observation by a conscious observer is essential to the understanding of quantum theory. Such a conclusion is without foundation, for we have seen that the Statistical Interpretation does not require any such considerations. Moreover, the atoms of the observer's body may, in principle, be treated by quantum theory, and the resulting state vector will be similar to (4.3) (with I replaced by I+II, and II replaced by III). Von Neumann's theory would require another hypothetical observer IV to observe III, and so *reduce* the state vector. But this can only lead to an infinite regression, which, in an earlier era might have been taken as a proof of the existence of God (the Ultimate Observer)! No one seems to have drawn this unfounded conclusion, but the equally unfounded conclusion that the consciousness of an observer should be essential to the theory has, unfortunately, been taken seriously (Heitler, 1949, p. 194; Wigner, 1962).

Because of the difficulties inherent in the projection postulate, we must critically examine the reasons given in support of its introduction. Dirac (1958, p. 36) argues that a second measurement of the same observable immediately after the first measurement must always yield the same result. Clearly this could be true at most for the very special class of measurements that do not change the quantity being measured. A statement of such limited applicability is hardly suitable to play any fundamental role in the foundations of quantum theory. In fact this argument is based on the implicit (and incorrect) assumption that *measurement* is equivalent to *state preparation*, the contrasting definitions of which were given in Sec. 3. The necessity for distinguishing between these two concepts has been pointed out by Margenau (1958; 1963). For example, a polaroid filter placed in the path of a photon beam constitutes a *state preparation* with respect to the polarization of any transmitted photons. A second polaroid at the same angle has no further effect. But neither of these processes constitutes a *measurement*. To measure the polarization of a photon one must also detect whether or not the photon was transmitted through the polaroid filter. Since the detector will absorb the photon, no second measurement is possible.

It has also been claimed (Messiah, 1964, p. 140) that a discontinuous *reduction* of the state vector is a consequence of an alleged uncontrollable perturbation of the object by the measuring apparatus. A counter example to this claim is provided by the EPR experiment (Sec. 2.1), in which the spin of particle 2 is

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[11] This was first pointed out by Schrödinger (1935), who proposed the following hypothetical experiment. A cat is placed in a chamber together with a bottle of cyanide, a radioactive atom, and a device which will break the bottle when the atom decays. One half-life later the state vector of the system will be a superposition containing equal parts of the living and dead cat, but any time we look into the chamber we will see either a live cat or a dead one. A literal translation of "Schrödinger's cat paradox", as this is called, has been given by Jauch (1968, p. 185), but his discussion of the paradox is not adequate, and his volume and page reference is incorrect.

[12] His conclusion that no theory consistent with quantum mechanics can account for the occurrence of events in nature, however, is not true if one drops the assumption that a state vector describes completely an individual system.

measured indirectly, without any disturbance of that particle whatsoever. In general the apparatus will interact with the object, but the effect of this is not such as to bring about the hypothetical *reduction*. We do not consider as credible the occasional statement (Dirac, 1958, p. 110) that a mere observation (i.e., an experimenter looking at his apparatus) could affect the state of the system in a manner incompatible with the equation of motion.

Although it is not essential to the theory, Popper (1967) has pointed out that the *reduced* state vector (4.4) can be interpreted in a natural way. According to the Statistical Interpretation, the state vector (4.3) represents, not an individual system, but the ensemble of all possible systems (object plus apparatus), each of which has been prepared in a certain way and then allowed to interact. The state vector (4.4) represents the subensemble whose definition includes the additional specification that the result of the measurement (pointer reading) be $\alpha_{r'}$. There is no question of *reduction of the state vector* being a physical process.

Margenau (1963, p. 476) has taken a more sympathetic attitude toward Von Neumann's theory of measurement on the grounds (gathered from personal conversation) that he regarded his projection postulate as convenient but not necessary. In reply I would say that I find no evidence of this interpretation in Von Neumann's writings. Moreover, the great amount of confusion generated, as well as the theoretical effort expended trying to explain the *reduction of the state vector*, suggest that the projection postulate has been much more of an inconvenience than a convenience.

4.3 Other Approaches to the Problem of Measurement

An enormous number of papers have been written on the problem of measurement in quantum theory. Margenau (1963), lists over 60 separate articles, and promises a list of about 200 on request. The reader can shorten his task greatly by ignoring all papers which try, without modifying quantum theory, to accomodate a *reduction of the state vector*, and which also assume the state vector to describe an individual system. The preceding arguments have demonstrated the impossibility of such programs.

Bohm and Bub (1966) propose a nonlinear modification of the Schrödinger equation of motion which is supposed to be effective only during the measurement process, and which causes the *reduction*. However we do not consider that the postulation of a new interaction peculiar to measurement should be taken seriously. Moreover, the simplicity of the account of measurement provided by the Statistical Interpretation undermines the motivation for any drastic modification of the mathematical formalism.

To discuss the next approach it is convenient to restate the results of our analysis in the language of state operators (or statistical operators). After the

interaction between the object I and the apparatus II, the final state operator for the combined system I+II will be the pure state

$$\rho_f = \mid I+II; f \rangle \langle I+II; f \mid, \qquad (4.5)$$

$$\mid I+II; f \rangle = \sum_r c_r \mid I; \phi_r \rangle \mid II; \alpha_r \rangle, \qquad (4.3')$$

where $c_r = \langle r \mid \psi \rangle$ in the notation of (4.3). Now the mixed state formed from *reduced* vectors of the type (4.4),

$$\rho_m = \sum_r \mid c_r \mid^2 \mid I; \phi_r \rangle \mid II; \alpha_r \rangle \langle I; \phi_r \mid \langle II; \alpha_r \mid \quad (4.6)$$

is not equivalent to ρ_f. Even though they both yield the same probability distribution for the position of the apparatus "pointer" α, their predictions for other observables may disagree.[13] Because time development is effected by a unitary transformation (1.7a), which preserves the pure state condition $\rho^2 = \rho$, it is impossible for the pure state (4.5) ever to evolve into the mixed state (4.6).

Realizing this fact, many authors (Feyerabend, 1957; Wakita, 1960, 1962; Daneri *et al.*, 1962; Jauch, 1964) have proposed that, nevertheless, (4.5) and (4.6) may be equivalent for all practical purposes. Specifically, they suggest that if the macroscopic nature of the apparatus and its relevant observables are taken into account, then the average of any *macroscopic* observable M will be the same for these two states. With the abbreviated notation $\mid \phi_r, \alpha \rangle \equiv \mid I; \phi_r \rangle \mid II; \alpha \rangle$, these averages are, respectively ,

$$\text{Tr } (\rho_f M) = \sum_{r,r'} c_{r'} c_r^* \langle \phi_r, \alpha_r \mid M \mid \phi_{r'}, \alpha_{r'} \rangle, \quad (4.7)$$

$$\text{Tr } (\rho_m M) = \sum_r \mid c_r \mid^2 \langle \phi_r, \alpha_r \mid M \mid \phi_r, \alpha_r \rangle. \qquad (4.8)$$

The problem is then whether these expressions are equal under sufficiently general circumstances.

Now if M commutes with A (the pointer position, whose eigenvalues are α_r), it will be diagonalizable in this representation, and the interference terms $(r \neq r')$ will be absent from (4.7). Hence the two expressions will be equal. However, (contrary to Jauch's assumption) the set of macroscopic observables is not Abelian (consider the pointer momentum, which does not commute with A), so the macroscopic nature of M is not sufficient to guarantee equality of (4.7) with (4.8).

In the case where the observable M belongs entirely

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[13] A proof of this fact is contained in a paper by Furry (1936). But his comments on the work of Einstein, Podolsky, and Rosen (Sec. 2) are based upon a misconception. Furry interprets the EPR conclusion that certain dynamical variables such as σ_{2z} in our Eq. (2.6) are *elements of reality*, as meaning that the state of the system after interaction between the object and the apparatus is a mixture of eigenstates of the *element of reality*; i.e , a mixed state related to (2.6) in the same way as (4.6) is related to (4.5). But EPR do not claim that the equation of motion yields the wrong quantum state, as Furry has misinterpreted them, but rather that no quantum state, pure or mixed, can provide a complete description of an individual system. This makes Furry's argument irrelevant.

to the measurement apparatus II (i.e., if $M = 1 \otimes M_{II}$),[14] then

$$\langle \phi_r, \alpha_r \mid M \mid \phi_{r'}, \alpha_{r'} \rangle$$
$$= \langle I; \phi_r \mid I; \phi_{r'} \rangle \langle II; \alpha_r \mid M_{II} \mid II; \alpha_{r'} \rangle. \quad (4.9)$$

Thus if the states $\mid I; \phi_r \rangle$ are mutually orthogonal, which would be true, in particular, if the measurement of R did not change the value of R, ($\mid I; \phi_r \rangle$ then becomes $\mid I; r \rangle$), then there would be no difference between (4.7) and (4.9). Clearly a similar result holds for observables belonging entirely to the object I, but of course no such result is valid for observables which belong to I and II jointly.[15]

Daneri, Loinger, and Prosperi (1962, 1966) (DLP) consider the pure state (4.5) or equivalently (4.3) to be the final state of the combined system *immediately* after interaction between the object and the apparatus. They then concentrate on the amplification aspect of the measurement necessary for a microscopic object to trigger a macroscopic response in the apparatus, and invoke the ergodic theory of quantum statistical mechanics in order to treat the transition of the apparatus from its initial "metastable"[16] condition to equilibrium. They claim, in essence, that the equilibrium state can be represented by the mixed state operator (4.6).

The attitude of DLP is well expressed in the words of Rosenfeld (1965, p. 230), who endorses their approach: "The reduction of the initial state of the atomic system has nothing to do with the interaction between this system and the measuring apparatus: in fact, it is related to a process taking place in the latter apparatus after all interaction with the atomic system has ceased." Rosenfeld also emphasizes the view that the presence or absence of a conscious human observer is irrelevant to the problem.[17]

The above remarks show that DLP and Rosenfeld explicitly reject the "orthodox" interpretation discussed in Sec. 4.2 (as does the present author), but they do so without actually confronting the question of whether the state operator (or vector) represents a single system or an ensemble of similarily prepared systems.

The analysis of generalized quantum amplification devices by DLP is interesting and correct, at least to the extent that their ergodic hypothesis is valid. But it does not constitute the essence of measurement in general. To paraphrase an example given by Jauch

(1968, p. 169), if the measuring device is a photographic plate, then the essence of measurement may be performed by a single silver–halide complex. The amplification process does not take place until the plate is developed, which may be months later. Although interesting in its own right, the DLP theory does not answer the questions which lead us to study the theory of measurement.

4.4 Conclusion—Theory of Measurement

This section on measurement theory is lengthy because of the great length of the literature on this subject. However, the conclusions are quite simple. The formal analysis of a measurement has been undertaken not for its own sake, but only to test the consistency of alternative interpretations of quantum theory. Using only the linearity of the equation of motion and the definition of measurement, we see that the interaction between the object and the measuring apparatus leads, in general, to a quantum state which is a coherent superposition of macroscopically distinct "pointer positions." In the Statistical Interpretation this dispersion of pointer positions merely represents the frequency distribution of the possible measurement results for a given state preparation.

But if one assumes that the state vector completely describes an individual system, then the dispersion must somehow be a property of the individual system, but it is nonsensical to suppose that a macroscopic pointer has no definite position. None of the attempts to solve this problem using some form of *reduction of the state vector* are satisfactory. No such problem arises in the Statistical Interpretation, in which the state vector represents an ensemble of similarily prepared systems, so it must be considered superior to its rivals.

We have neglected the frankly subjective interpretation (Sussmann, 1957; Heisenberg, 1958), according to which the quantum state description is not supposed to express the properties of a physical system or ensemble of systems but our *knowledge* of these properties, and changes of the state (such as the supposed reduction of state during measurement) are identified with changes of knowledge, not so much because it is wrong as because it is irrelevant.[18] One can, of course, study a person's knowledge of physics rather than studying physics itself, but such a study is not germane to this paper. For example, someone's knowledge of a certain system can change discontinuously as a result of a blow to the head which causes amnesia, as well as through the receipt of new information from a

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[14] The tensor product notation $A \otimes B$ means that A operates on the $\mid I \rangle$ vectors, and B operates on the $\mid II \rangle$ vectors.

[15] This illustrates the fact that the state of a two-component system cannot be uniquely determined by measurements on each component separately. Jauch (1968, Secs. 11-7 and 8) gives a good discussion of the mathematical aspects of this problem, and in particular of the tensor product formalism.

[16] As pointed out in DLP (1966), this is not necessarily the thermodynamic definition of metastability.

[17] On this point Rosenfeld seems to have modified his earlier views, as expressed in a discussion exchange with Viger (see Körner, 1957, pp. 183–6).

[18] Another area in which the subjective interpretation causes confusion is the relationship between entropy and information. It is true that the aquisition of information requires a corresponding increase of entropy (Brillouin, 1962); hence the association of information with negative entropy is useful. But the converse proposition that entropy is nothing but a measure of our lack of information is a fallacy. Entropy also has a thermodynamic meaning, $dS = dQ/T$, which is valid regardless of the existence or nonexistence of Maxwell's demon.

measurement, although the subjectivists tend to ignore the former while stressing the latter.

5. JOINT PROBABILITY DISTRIBUTIONS

In contrast to the previous sections, which were concerned entirely with interpretation of the existing formalism (summarized in Sec. 1.1), this section and the next consider possible extensions of the formalism. These will not be modifications of the established formalism, but additions to it which are compatible with both the established formalism and the Statistical Interpretation.

As discussed in Sec. 3 with regard to the uncertainty principle, quantum theory is not inconsistent with the supposition that a particle has at any instant both a definite position and a definite momentum, although there is a widespread folklore to the contrary. Of course we are not compelled either to accept or to reject this supposition, but it is of interest to explore it on a tentative basis.

Our first problem is to construct, for the noncommuting observables q and p, a joint probability distribution $P(q, p; \psi)$ for any state $| \psi \rangle$,[19] such that the single variable distributions constructed from it agree with the established formalism, i.e.,

$$\int P(q, p; \psi) \, dp \equiv P(q; \psi) = |\langle \psi \mid q \rangle|^2, \quad (5.1)$$

$$\int P(q, p; \psi) \, dq \equiv P(p; \psi) = |\langle \psi \mid p \rangle|^2. \quad (5.2)$$

Additional conditions will be considered later.

Let us first consider the method by which the single observable probability distribution function can be constructed. The characteristic function for the distribution of an arbitrary observable A is given by

$$M(\xi; \psi) = \langle \exp (i\xi A) \rangle$$
$$= \int \exp (i\xi A) P(A; \psi) \, dA, \quad (5.3)$$

hence $P(A; \psi)$ is equal to the inverse Fourier transform of $M(\xi; \psi)$. By expanding the exponential

$$M(\xi; \psi) = \sum_{n=0}^{\infty} \frac{(i\xi)^n}{n!} \langle A^n \rangle, \quad (5.4)$$

we see that a knowledge of the characteristic function is equivalent to a knowledge of all the moments of the distribution. According to the formalism (see F4), these are given by[20]

$$\langle A^n \rangle = \langle \psi \mid \hat{A}^n \mid \psi \rangle. \quad (5.5)$$

By analogy we may introduce a characteristic function for the joint probability distribution

$$M(\theta, \lambda; \psi) = \sum_{n,m=0}^{\infty} \frac{(i\theta)^n (i\lambda)^m}{n!m!} \langle q^n p^m \rangle, \quad (5.6)$$

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[19] We shall consider only pure states since these illustrate the essential problems. The generalization to mixed states is straightforward.

[20] In this section it is necessary to distinguish between a physical observable and the mathematical operator which represents it. The operators will be distinguished by a circumflex.

from which, by a Fourier inversion, we obtain,

$$P(q, p; \psi)$$
$$= (2\pi)^{-2} \int \int M(\theta, \lambda; \psi) \exp [-i(\theta q + \lambda p)] \, d\theta d\lambda. \quad (5.7)$$

As long as the moments $\langle q^n \rangle$ and $\langle p^m \rangle$ are given by (5.5), then (5.7) will automatically satisfy (5.1) and (5.2).

The difficulty now evident is that that there is no unique way in which to take products of noncommuting operators. For example $q^2 p^2$ may be represented by $\frac{1}{2}(\hat{q}^2\hat{p}^2 + \hat{p}^2\hat{q}^2)$, $\frac{1}{4}(\hat{q}\hat{p} + \hat{p}\hat{q})^2$, $\frac{1}{2}(\hat{q}\hat{p}\hat{q}\hat{p} + \hat{p}\hat{q}\hat{p}\hat{q})$, or any of several other forms. In these examples we have already used the additional plausible restrictions that the product be a self-adjoint operator, and that it be symmetric under interchange of $\hat{q}$ and $\hat{p}$. Shewell (1959) has considered several correspondence rules which have been proposed, and found none to be fully satisfactory. Some of the rules do not yield a unique operator, while others yield an operator for a power of the Hamiltonian, $[H(q, p)]^n$, which is not equal to $(\hat{H})^n$. Each such possible choice will yield a different joint probability distribution which satisfies (5.1) and (5.2).

If one chooses

$$\exp [i(\theta q + \lambda p)] \rightarrow \exp (i\theta\hat{q} + i\lambda\hat{p}), \quad (5.8a)$$

or equivalently

$$q^n p^m \rightarrow 2^{-n} \sum_{l=0}^{n} \binom{n}{l} \hat{q}^{n-l} \hat{p}^m \hat{q}^l, \quad (5.8b)$$

then one obtains (Moyal, 1949)

$$P(q, p; \psi)$$
$$= (2\pi)^{-1} \int \langle \psi \mid q - \tfrac{1}{2}\tau\hbar \rangle \exp (-i\tau p) \langle q + \tfrac{1}{2}\tau\hbar \mid \psi \rangle \, d\tau, \quad (5.9)$$

which was first introduced by Wigner (1932). Because this expression may become negative it cannot be interpreted as a genuine probability distribution, and Wigner proposed it only as a calculational device. Another distribution which suffers from the same defect has been derived by Margenau and Hill (1961) using the correspondence

$$q^n p^m \rightarrow \tfrac{1}{2}(\hat{q}^n\hat{p}^m + \hat{p}^m\hat{q}^n). \quad (5.10)$$

Since every correspondence rule which associates an operator with a classical function of q and p,

$$g(q, p) \rightarrow \hat{G}(\hat{q}, \hat{p}), \quad (5.11)$$

provides a joint probability distribution, it is possible to formulate the most general form of $P(q, p; \psi)$ which will satisfy (5.1) and (5.2). We refer to the original papers for details (Cohen, 1966; Margenau and Cohen, 1967). These authors investigated the possibility of constructing a joint probability distribution such that

$$\langle \psi \mid \hat{G}(\hat{q}, \hat{p}) \mid \psi \rangle = \int \int g(q, p) P(q, p; \psi) \, dq dp, \quad (5.12)$$

and also such that for any function K

$$\langle \psi \mid K(\hat{G}(\hat{q}, \hat{p})) \mid \psi \rangle = \iint K(g(q, p)) P(q, p; \psi) \, dq dp.$$

$$(5.13)$$

In other words, they asked whether there exists a joint probability distribution such that the averages of all observables can be calculated by a phase-space average as in classical statistical mechanics.[21]

Their answer was negative. Although it is always possible to satisfy (5.12), (5.13) could be simultaneously satisfied only if the correspondence

$$K(g(q, p)) \rightarrow K(\hat{G}(\hat{q}, \hat{p}))$$

$$(5.14)$$

can be satisfied for all functions K with the same correspondence rule as that leading to (5.11). This they prove to be impossible.[22] With the wisdom of hindsight, we should not be surprised that (5.12) and (5.13) cannot be satisfied, for if it were otherwise, then quantum mechanics could be expressed as a special case of classical mechanics.

To deal with simultaneous values of position and momentum variables in a logically consistent fashion we need only require that a joint probability distribution exists which satisfies (5.1) and (5.2), and

$$P(q, p; \psi) \geq 0$$

$$(5.15)$$

for all quantum states. These conditions are obviously satisfied by the function

$$P(q, p; \psi) = \mid \langle \psi \mid q \rangle \mid^2 \mid \langle \psi \mid p \rangle \mid^2,$$

$$(5.16)$$

but this form is probably not unique. Indeed, it cannot be applicable to scattering experiments (see Fig. 3), where, merely by geometry, there must be a close correlation between position and momentum at large distances from the scattering center. The investigation of possible joint probability distributions is clearly not complete.

Bopp (1956, 1957) has undertaken not only to represent quantum mechanics in terms of ensembles in phase space, but also to derive the statistical equations from simple principles—an *a priori* derivation—not merely a translation from the vector-operator formalism of quantum theory. Bopp's work differs from the above in that his joint probability distribution does not satisfy (5.1) and (5.2). Nevertheless, there is a definite relation between the averages calculated in Bopp's theory and those of quantum theory (denoted by subscripts B and Q). For example,

$$\langle q \rangle_B = \langle q \rangle_Q, \qquad \langle p \rangle_B = \langle p \rangle_Q,$$
$$\langle q^2 \rangle_B = \langle q^2 \rangle_Q + l^2/4,$$
$$\langle p^2 \rangle_B = \langle p^2 \rangle_Q + \hbar^2/l^2.$$

Here l is some constant with dimensions of length (interpreted by Bopp as the finest accuracy of a position measurement).

The problem of defining a satisfactory joint probability distribution for position and momentum may not be entirely mathematical, for we must also decide on "the empirical definition of the concept 'simultaneous values of position and momentum,'" according to Prugovecki (1967). To this end he studies various operational methods of measuring both these observables *simultaneously* on the same individual system, not merely on different individual systems representative of the same state preparation. He suggests that the irreducible errors in individual measurements ought to be taken into account in a generalized formalism (in addition to the statistical fluctuations in an ensemble of similar measurements, which are treated in the established formalism). It should be emphasized that simultaneous measurement of noncommuting observables, even with a finite precision, has no counterpart in the established formalism which treats only measurement of a single observable (or a commuting set of observables).[23] Now to experimentally determine a probability distribution, one must perform a number of measurements and construct a histogram of the results (see Fig. 2). Each point on the curve will have a vertical uncertainty due to the statistical uncertainty of using a finite sample, and a horizontal uncertainty due to the error involved in a single measurement. (In the case of simultaneous measurement of q and p, a single measurement would be a pair of numbers (q, p), and the error would be an area in the qp plane). If the latter cannot be reduced to zero, we cannot determine a probability distribution exactly, even if we eliminate statistical uncertainties by repeating the measurement an arbitrarily large number of times.

To deal with such a circumstance Prugovecki defines a *complex probability distribution*, $P_\psi{}^{q,p}(I_1 \times I_2)$, with the following properties: the real part of P represents the probability that q will be in the interval, I_1 and simultaneously p will be in the interval I_2, where

$$I_1 = \{\lambda : q_1 \leq \lambda \leq q_2\},$$
$$I_2 = \{\mu : p_1 \leq \mu \leq p_2\},$$

$$(5.17)$$

and the imaginary part of P represents the uncertainty associated with the previous statement. Certain conditions must be satisfied in order for this interpretation to be sensible. Since q and p are certain to be somewhere in their spectra, we must have Re $P \rightarrow 1$, and Im $P \rightarrow 0$, as I_1 and I_2 are enlarged to include the entire spectra from $-\infty$ to $+\infty$. If I_1 and I_2 are smaller than the irreducible errors in a single measurement,

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[21] Because (5.1) and (5.2) are satisfied, this will clearly be true for any function of q only, or of p only.

[22] It would seem inevitable that arbitrary functional relations involving q and p cannot be preserved by any quantal operator correspondence rule, since the equation $qp - pq = 0$ for classical variables becomes $\hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar$ in quantum theory. Clearly one cannot map zero onto $i\hbar$ in a consistent fashion.

[23] The naive interpretation (Matthews, 1968, Chap. 3) of the operator product AB as corresponding to a measurement of B followed by a measurement of A is refuted by the example of the Pauli spin operators for which $\sigma_x \sigma_y = i\sigma_z$. A measurement of σ_y followed by a measurement of σ_x is in no way equivalent to a measurement of σ_z.

then Im P should be very large because one cannot determine whether or not a measured value of (q, p) lies within such a small region of phase space, and therefore one cannot determine the relative frequency (probability) of such an event. Of course the complex probability distribution must reduce to the real probability distribution of the established formalism in the case of commuting operators.

The specific expression proposed by Prugovecki for the complex probability distribution is

$$P_\psi{}^{q,p}(I_1 \times I_2) = \langle \psi \mid E_1(I_1)E_2(I_2) \mid \psi \rangle, \quad (5.18)$$

where E_1 and E_2 are projection operators which project onto the subspaces spanned by the eigenvectors of $\hat{q}$ corresponding to the spectral interval I_1, and of $\hat{p}$ corresponding to I_2, respectively. This form, although perhaps the most obvious choice, may not be satisfactory because he shows that for any nontrivial intervals I_1 and I_2 there exist states for which Re $P<0$. This would be tolerable if simultaneously we had a large value for Im P, so that we could say that the probability was not well defined in such a case. But in fact one obtains Im $P=0$ for that state which makes Re P most negative. Since a "well defined" but negative probability makes no sense, some modification of either (5.18) or the interpretation seems in order (such as, perhaps, a restriction on the states ψ which are to be considered physically realizable). This objection does not necessarily invalidate the concept of a complex probability distribution, but the usefulness of such a distribution is yet to be determined.

6. HIDDEN VARIABLES

Quantum theory predicts the statistical distribution of events (i.e., the results of similar measurements preceded by a certain state preparation). But if the prepared state does not correspond to an eigenvector of the particular observable being measured, then the outcome of any individual event is not determined by quantum theory. Thus one is led to the conjecture that the outcome of an individual event may be determined by some variables which are not described by quantum theory, and which are not controllable in the state preparation procedure. The statistical distributions of quantum theory would then be averages over these "hidden variables."

The entirely reasonable question, "Are there hidden-variable theories consistent with quantum theory, and if so, what are their characteristics?," has been unfortunately clouded by emotionalism. A discussion of the historical and psychological origins of this attitude would not be useful here. We shall only quote one example of an argument which is in no way extreme (Inglis, 1961, p. 4), "Quantum mechanics is so broadly successful and convincing that the quest [for hidden variables] does not seem hopeful." The vacuous character of this argument should be apparent, for the success

of quantum theory within its domain of definition (i.e., the calculation of statistical distributions of events) has no bearing on the existence of a broader theory (i.e., one which could predict individual events).

The question of hidden variables has also been subject to a genuine confusion (i.e., not merely due to the conflicting personalities or metaphysical ideas of the principal characters in the debate), for Von Neumann proved a theorem from which he concluded that no hidden-variable theory could reproduce all of the statistical predictions of quantum theory. Although his mathematical theorem is correct, his physical conclusion is not, and the first example of a hidden-variable model was published by Bohm (1952). However, a clear analysis of the nature of Von Neumann's theorem and its relation to hidden variables was not achieved until much later (Bell, 1966), and the 14 year interval between these papers yielded many inconclusive discussions pro and con, as well as some "improved proofs" of the impossibility of hidden variables. We shall expound briefly the content of Von Neumann's theorem, and the reason why it does not rule out hidden variables in quantum theory. Other relevant theorems will also be discussed.

The Statistical Interpretation, which regards quantum states as being descriptive of ensembles of similarily prepared systems, is completely open with respect to hidden variables. It does not demand them, but it makes the search for them entirely reasonable [this was the attitude of Einstein (1949)]. On the other hand, the Copenhagen Interpretation, which regards a state vector as an exhaustive description of an individual system, is antipathetic to the idea of hidden variables, since a more complete description than that provided by a state vector would contradict that interpretation.

6.1 Von Neumann's Theorem

Von Neumann's mathematical theorem concerns the average of an observable, represented by a Hermitian operator, in a general ensemble subject only to a few conditions. The original derivation (Von Neumann, 1955, pp. 305–325) is lengthier than necessary, and we shall follow the more concise version given by Albertson (1961). Von Neumann makes the following essential assumptions which in his book are identified by the symbols in brackets:

(i) [I] If an observable is represented by the operator R, then a function f of that observable is represented by $f(R)$.

(ii) [II] The sum of several observables represented by $R, S, \cdots$ is represented by the operator $R+S+\cdots$, regardless of whether these operators are mutually commutative.

(iii) [p. 313, not identified] The correspondence between Hermitian operators and observables is one to one.

(iv) [A'] If the observable R is nonnegative, then $\langle R \rangle \geq 0$.

(v) [B'] For arbitrary observables R, S, $\cdots$ and arbitrary real numbers a, $b \cdots$, we must have

$$\langle aR + bS + \cdots \rangle = a\langle R \rangle + b\langle S \rangle + \cdots \quad (6.1)$$

for all possible ensembles (or states) in which the averages may be calculated.

An arbitrary Hermitian operator R may be written as

$$R = \sum_{m,n} | n \rangle R_{nm} \langle m |$$

$$= \sum_n U^{(n)} R_{nn}$$

$$+ \sum_{m,n; m<n} \{ V^{(nm)} \operatorname{Re} R_{mn} + W^{(nm)} \operatorname{Im} R_{mn} \}, \quad (6.2)$$

where $R_{nm} = R_{mn}^* = \langle n | R | m \rangle$ is a number, and

$$U^{(n)} = | n \rangle \langle n |,$$

$$V^{(nm)} = | n \rangle \langle m | + | m \rangle \langle n |,$$

$$W^{(nm)} = -i(| n \rangle \langle m | - | m \rangle \langle n |) \quad (6.3)$$

are Hermitian operators. According to assumption (iii) these operators are all to be regarded as observables, and hence (ii) and (v) may be applied to obtain

$$\langle R \rangle = \sum_n \langle U^{(n)} \rangle R_{nn}$$

$$+ \sum_{m,n; m<n} \{ \langle V^{(nm)} \rangle \operatorname{Re} R_{mn} + \langle W^{(nm)} \rangle \operatorname{Im} R_{mn}. \quad (6.4)$$

This result may be rewritten as

$$\langle R \rangle = \sum_{m,n} \rho_{nm} R_{mn} = \operatorname{Tr}(\rho R), \quad (6.5)$$

if we define the matrix elements of the statistical operator ρ to be

$$\rho_{nn} = \langle U^{(n)} \rangle,$$

$$\rho_{mn} = \tfrac{1}{2} \langle V^{(nm)} \rangle + \tfrac{1}{2} i \langle W^{(nm)} \rangle \quad (m<n),$$

$$\rho_{nm} = \tfrac{1}{2} \langle V^{(nm)} \rangle - \tfrac{1}{2} i \langle W^{(nm)} \rangle \quad (m<n). \quad (6.6)$$

Since the averages of observables are real, the operator ρ is Hermitian by construction, and assumption (iv) implies that it is nonnegative definite ($\langle \phi | \rho | \phi \rangle \geq 0$ for all ϕ).

This completes the theorem of Von Neumann, which may be summarized by saying that the statistical operator representation of states (F4 of our Sec. 1.1) need not be introduced as a postulate, but may instead be derived from assumptions (i) $\cdots$ (v). This theorem, which is clearly of interest outside of the question of hidden variables, is discussed in more detail in Chapter 5 of Jordan (1969).

Hidden variables are, by definition, hypothetical variables whose values must be specified in addition to the values of a complete commuting set of observables in order to uniquely determine the result of any measurement on a system. Quantum states, with their characteristic statistical distributions, would represent

ensembles of systems with different values of the hidden variables, but if all hidden variables were fixed, the resultant subquantal ensemble would be dispersionless. That is,

$$\langle (R - \langle R \rangle)^2 \rangle = 0, \quad (6.7a)$$

or equivalently

$$\langle R^2 \rangle = \langle R \rangle^2, \quad (6.7b)$$

for all observables R, since by hypothesis every quantity would have a unique value. Applying the result (6.5) to (6.7b) for the case of $R = | \phi \rangle \langle \phi |$, where $\langle \phi | \phi \rangle = 1$, yields

$$\langle \phi | \rho | \phi \rangle = \langle \phi | \rho | \phi \rangle^2 \quad (6.8)$$

for all normalized vectors $| \phi \rangle$. This implies $\langle \phi | \rho | \phi \rangle = 1$ or 0 for all $| \phi \rangle$, and since this expression varies continuously with $| \phi \rangle$, the constant—1 or 0—must be the same for all $| \phi \rangle$. Hence we must have either

$$\rho = 1 \quad \text{or} \quad \rho = 0 \quad (6.9)$$

for dispersion-free ensembles.

The case $\rho = 0$ is ruled out because it would imply $\langle R \rangle = 0$ for all R in any subquantal ensemble, and so averages of these could never yield the correct averages in quantum ensembles. The case $\rho = 1$ does not in fact yield a dispersionless ensemble if the vector space is greater than one dimensional. In this case

$$\langle 1 \rangle = \operatorname{Tr}(1) = d, \quad (6.10)$$

d being the dimensionality of the space (usually ∞), and the left-hand side of (6.7a) becomes

$$\langle R^2 \rangle - 2\langle R \rangle^2 + \langle R \rangle^2 \langle 1 \rangle \quad (6.11)$$

which is not zero. Thus Von Neumann concludes that, provided his assumptions (i)–(v) are accepted, there are no dispersion-free ensembles, and hence there can be no hidden variables. Since these assumptions are generally considered to be a part of quantum theory, he states "It is therefore not, as is often assumed, a question of a reinterpretation of quantum mechanics,—the present system of quantum mechanics would have to be objectively false $\cdots$," in order for even hypothetical hidden variables to be introduced in a logically consistent fashion.

The above conclusion is incorrect, but before we discuss the reasons why, let us dispose of two minor points of confusion. Albertson (1961, 1962) makes misleading and incorrect statements to the effect that Von Neumann's theorem does not *assume* the existence of noncommuting observables, and that the theorem is independent of, and additional to, the uncertainty principle. But the assumption (iii), which Von Neumann did not distinguish by any symbol, in effect assumes the existence of an infinite number of noncommuting observables, and this assumption plays a very central role in the proof.[24] If one assumed that all observables

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[24] Oddly enough, Jammer (1966) repeats Albertson's incorrect statement on p. 369, while on p. 370 he mentions Von Neumann's assumption that every projection operator is an observable, without being aware of the contradiction!

were commutative, then a simultaneous eigenvector of all the observables would represent a dispersion-free ensemble.

The second minor point concerns the observation that (iii) is trivially false if there are superselection rules, which divide the full Hilbert space into coherent subspaces such that no observable may have matrix elements between vectors of different subspaces and no physical state vector may be a superposition of vectors from different subspaces (Wick *et al.*, 1952; Galindo *et al.*, 1962). But Von Neumann's theorem can still be proven in each coherent subspace individually, so the question of superselection rules is irrelevant.

6.2 Bell's Rebuttal

There is nothing wrong with the mathematics of Von Neumann's theorem. Nevertheless his conclusion that no hidden-variable model can be consistent with the statistical predictions of quantum theory is false— for such models exist (Bohm, 1952). The difficulty lies in the relation between the mathematics and the physics, as was clearly analyzed by Bell (1966).

The result of Von Neumann's theorem [Eq. (6.5)] seems to imply that any ensemble can be characterized by a statistical operator, whether this ensemble is a quantum state or a subquantal hidden-variable state. But since we know that a statistical operator characterizes just a general quantum state, we should immediately become suspicious of the assumptions which lead to this result. As Bell pointed out, it is (6.1) of assumption (v) which is at fault. At first sight the requirement that the average of the sum of two observables be equal to the sum of the averages of the observables separately may seem reasonable. But the nontrivial nature of this additivity property becomes apparent when we realize that it is not true for individual measurements of noncommuting observables. Consider for example the spin components of a particle. The measurement of σ_x can be made with a suitably oriented Stern–Gerlach magnet. The measurement of σ_y requires another orientation. There is no way of relating a measurement of $(\sigma_x+\sigma_y)$ to the results of the first two measurements. This requires a third and different orientation of the magnet. Moreover, the result of the measurement, an eigenvalue of $(\sigma_x+\sigma_y)$, will *not* be the sum of an eigenvalue of σ_x plus an eigenvalue of σ_y. That the ensemble average of many measurements of $(\sigma_x+\sigma_y)$, $\langle(\sigma_x+\sigma_y)\rangle$, should be equal to the sum of the averages of two other measurements involving different experimental arrangements, $\langle\sigma_x\rangle+\langle\sigma_y\rangle$, is a peculiar and nontrivial property of quantum states. But in a hypothetical dispersion-free state every observable would have a unique value equal to one of its eigenvalues, and since there is no linear relationship between the eigenvalues of noncommuting observables, in general, it is obvious that (6.1) could not possibly be satisfied for dispersion-free states. When this impossible condition is removed it is possible

to construct a hidden-variable model which reproduces the correct statistical distributions for quantum states, as Bell showed by means of a simple example.

Bell also considers the relevance of some more recent mathematical work by Jauch and Piron (1968) and by Gleason (1957) to the problem of hidden variables, and we refer the reader to his paper for very clear discussions. Although the mathematical theorems of Von Neumann and the above authors do not in fact exclude hidden variables from quantum theory, they are not entirely devoid of value. With proper analysis, such as Bell provided, they help to point out some of the features which a successful hidden-variable model must possess.

The work of Gleason (1957) is particularly interesting, for he shows that the additivity assumption (6.1) for *commuting observables only* is incompatible with the requirement that every projection operator have a unique value (either 0 or 1) in a dispersion-free state. Bell shows that one may still introduce hidden variables by invoking the dependence of a measurement result on the "whole experimental arrangement" (Bohr, 1949, p. 222) in a very strong way. In his model the particular result obtained when measuring a certain observable may depend upon which other commuting observables are being measured simultaneously.

One undesirable feature of Gleason's work, in common with that of Von Neumann, is the assumption that every projection operator represents an observable. Since every Hermitian operator is a linear combination of commuting projection operators, this is essentially equivalent to the assumption that every Hermitian operator represents an observable (see F7 of Sec 1.1). We are not concerned with the comparatively trivial restrictions on observability imposed by superselection rules, but with the question of whether an operator such as $x^2p_x{}^3x^2$ really represents an observable quantity. One might conjecture, as this author has, that the restrictions Gleason's theorem imposes on hidden variables might be met in a less drastic fashion than that proposed by Bell, if only essential observables, rather than all projection operators, were required to have unique values (eigenvalues) in a hypothetical dispersion-free state. The problem of enumerating "essential observables" has not been seriously considered, but an important example due to Bell (1964), to be discussed, indicates that such a proposal would not be adequate.

It seems appropriate to reply here to Jauch and Piron (1968) and Misra (1967), who have commented somewhat unfavorably on the work of Bell (1966). Misra (p. 845) states, "The quest for hidden variables becomes a meaningful scientific pursuit only if states, even physically unrealizable states, are somehow restricted by physical condition, ···" with the implication that they have done this but that Bell's analysis is only "a drastic *ad hoc* modification of the notion of state" (Jauch and Piron, 1968). While agreeing with

the first quoted statement above, I would contend that Misra's abstract algebraic approach, and the proposition-lattice theoretic approach of Jauch and Piron, fail to meet this requirement in that they impose only *abstract mathematical* conditions whose physical implications are obscured by their abstractness. The danger inherent in such an approach is underlined by the 34 year interval between Von Neumann's abstract attack on the hidden-variable problem and Bell's demonstration that one of his assumptions was physically unreasonable, and even impossible.

6.3 Bell's Theorem

Although hidden-variable models which reproduce all the statistical predictions of quantum theory are known to exist, they would be more interesting if they could be made to satisfy the very reasonable assumption (Einstein, 1949, p. 85) that the real factual situation of a system is independent of what is done with some other system which is spacially separated from, and not interacting with, the former. Bell (1964)[25] has considered this problem for the experiment which we discussed in connection with the EPR theorem, the measurement of arbitrary components of the spins of two, spin one-half particles, which were initially prepared in a singlet spin state, and which have separated.

The spin components $\sigma_1 \cdot a$ and $\sigma_2 \cdot b$ can both be measured by means of two Stern–Gerlach magnets oriented along the directions of the unit vectors a and b. The results of these two measurements will exhibit a statistical correlation, which for the singlet state is given by

$$\langle (\sigma_1 \cdot a)(\sigma_2 \cdot b) \rangle = -a \cdot b, \qquad (6.12)$$

according to quantum theory.

We now suppose that the results of individual measurements, which are not determined by the quantum state, are determined by some set of parameters denoted λ. The result A of measuring $\sigma_1 \cdot a$ is determined by a and λ, and the result B of measuring $\sigma_2 \cdot b$ is determined by b and λ; these results necessarily being eigenvalues of the operators, i.e.,

$$A(a, \lambda) = \pm 1, \qquad B(b, \lambda) = \pm 1. \qquad (6.13)$$

But in accordance with Einstein's assumption, the result A of measurement on particle 1 should not depend on the direction b of the magnet which acts on particle 2, and B should not depend on a. If $\rho(\lambda)$ is the probability distribution of the hidden variables λ, then the average of the product $(\sigma_1 \cdot a)(\sigma_2 \cdot b)$ will be given by

$$P(a, b) = \int A(a, \lambda) B(b, \lambda) \rho(\lambda) \, d\lambda. \qquad (6.14)$$

Bell then proves that (6.14) cannot be arbitrarily

close to the result of quantum theory (6.12). To avoid the possibility that the differences might occur only at isolated points, he first averages (6.12) and (6.14) over small cones of angles about the mean directions of a and b. We may write

$$|\overline{a \cdot b} - a \cdot b| \le \delta, \qquad (6.15)$$

where the bar denotes a smoothed function, with δ tending to zero with size of the range of angular averaging. Suppose that the difference between the smoothed functions is bounded,

$$|\bar{P}(a, b) + \overline{a \cdot b}| \le \epsilon. \qquad (6.16)$$

From a straightforward argument based on (6.13)–(6.16), Bell deduces the inequality

$$4(\epsilon + \delta) \ge \sqrt{2} - 1, \qquad (6.17)$$

from which it follows that as the range of angular smoothing, and hence δ, is made arbitrarily small, ϵ cannot be arbitrarily small. Hence (6.14) cannot be an arbitrarily accurate approximation to the quantum theory result (6.12).

Bell shows that it is easy to make a hidden-variable model to agree with (6.12) if the result $A(a, \lambda)$ of the measurement on particle 1 is allowed to depend also upon the direction b of the other magnet. However, if Einstein's hypothesis of independence of separated noninteracting systems is accepted, then the above theorem demonstrates that no such hidden-variable model can agree with all the predictions of quantum theory. This result can be interpreted as an illustration of Gleason's theorem as analysed by Bell (1966), according to which a hidden-variable model is possible only if the result of a measurement of $\sigma_1 \cdot a$ depends upon which one of the observables $\sigma_2 \cdot b$ (out of all the possible directions b) is being measured simultaneously. This example appears to rule out the possibility, conjectured in the previous section, that one might avoid the consequences of Gleason's theorem by narrowing one's attention from all Hermitian operators to only "essential observables," for there seems no doubt that $\sigma_1 \cdot a$ and $\sigma_2 \cdot b$ for all a and b are genuine observables.

Note added in proof: Recently Wigner (1970) has given a somewhat simpler version of Bell's argument, which yields some insight into its mathematical nature.

We shall not discuss specific hidden-variable theories in detail because none seems to be at all definitive, and any that reproduce *all* the results of quantum theory *exactly* must, as a consequence of Bell's theorem, be physically unreasonable (though they may be logically self-consistent). This remark would not apply to theories which depart from the formalism of quantum theory and only agree with it approximately in some limit. For descriptions of specific hidden-variable theories we refer the reader to a review by Freistadt (1957), and an article by Bohm (1962).

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[25] In spite of the publication dates, it is clear from the contexts that this paper was written later than Bell (1966).

6.4 Suggested Experiments

As emphasized by Bohm, the postulates of quantum theory can be tested only if we consider what it would mean to contradict them. Therefore, at the very least, the study of hidden variables will have served a useful purpose if it leads to the suggestion of interesting experiments, regardless of whether or not the outcome of those experiments is favorable to the hidden-variable hypothesis.

The additivity of averages of noncommuting observables in quantum ensembles (6.1) is a very powerful assumption, as Von Neumann's theorem shows, since it and a few other postulates allow one to derive (6.5) from which all the statistical predictions of quantum theory flow. Von Neumann (1955, p. 309, Footnote 164) suggested an experiment based upon the fact that the Hamiltonian operator is the sum of two noncommuting terms, the kinetic and potential energies. For some suitable state, one should measure the average kinetic energy by measurements of the momentum, measure the average potential energy by measurements of position, and measure the average energy by spectroscopic methods (each measurement being preceded by the same state preparation). Then the additivity of the quantum averages (expectation values) could be directly tested. A similar experiment which would probably be easier to perform could be based on the spin components σ_x, σ_y, and $(\sigma_x + \sigma_y) = \sqrt{2}$ (component of σ at $45°$ to the x and y axes).

The experiment involving the correlation of spin components of two particles from an initial singlet state is of interest because Bell (1964) showed that, while it is easy to construct a hidden-variable theory to reproduce the results of quantum theory for measurements on a *single* spin, it is not possible to do so for two-spin correlations without violating Einstein's assumption. Moreover, from (6.17) we see that the hidden-variable prediction must differ from the result of quantum theory by at least $\epsilon > 0.10$, so a clear cut experimental test is possible.

A measurement of the correlation in the polarizations of photons emitted from the decay of singlet positronium, which is similar in principle to the experiment above, has been performed by Wu and Shaknov (1950), but only for two angles. As Bell (1964) has pointed out, it is easy to make a hidden-variable theory agree with quantum theory at certain fixed angles, but not at all angles.

The hidden-variable theory of Bohm and Bub (1966) has been tested by Papaliolios (1967). According to the theory, when a photon from an arbitrarily linearly polarized beam is incident on a Polaroid filter (whose polarization axis is not exactly parallel to the polarization vector of the beam), the photon is or is not transmitted depending upon the value of some hidden variable. In a normal beam, the hidden variables are uniformly distributed, yielding the usual transmission probability. In the experiment, the beam is first passed through two nearly crossed polaroid filters. According to the theory, the transmitted beam should then have almost unique values for the hidden variables. These are presumed to relax once more to a uniform distribution within some relaxation time τ, but if the beam is incident on a third rotatable polarizer (whose axis is at a variable angle θ to the axis of the second), before this relaxation takes place the intensity transmitted through it is predicted to deviate from the $\cos^2 \theta$ law in a definite fashion. In the experiment, the third polarizer was placed as close to the second as possible, but no deviations from the $\cos^2 \theta$ law were observed. Papaliolios concludes that the hidden variables, if they exist, must relax to a uniform distribution in a time $\tau < 2.4 \times 10^{-14}$ sec, which is two orders of magnitude smaller than the value suggested by Bohm and Bub.

7. CONCLUDING REMARKS

The central theme connecting all of the topics in this paper has been the superiority of a purely statistical interpretation of quantum theory, in which a state vector represents an *ensemble* of similarly prepared systems, as opposed to the stronger assumption that a state vector provides a complete description of an *individual* system. By the criterion of logical economy, the Statistical Interpretation is preferable because it makes fewer assumptions. The stronger assumption plays no role in any application of quantum theory, and so Occam's razor may be invoked to discard it.

A more serious argument against the stronger assumption above is that it leads to conceptual difficulties. One such difficulty is exhibited by the Theorem of Einstein, Podolsky, and Rosen (Sec. 2) which demonstrates that this assumption is incompatible with the physical independence of spacially separated objects which may have interacted in the past. Although from a purely logical point of view one could retain either of these assumptions, it seems most unreasonable to discard the independence of separated noninteracting objects.

It used to be argued that the peculiarities consequent upon the assumption that a system is completely characterized by a wave function were confined to the microscopic domain, and that these peculiarities were justified by "the unavoidable disturbance of the system by the measuring apparatus." The latter of these contentions is refuted by the example due to EPR, in which it is possible to measure some observable of an elementary particle, without the apparatus interacting with that particle in any way. It appears (see Witmer, 1967, p. 45) that Bohr recognized this implication of the EPR argument, for he subsequently cautioned against such phrases as "disturbing of phenomena by observation" (Bohr, 1949, p. 237), but this lesson has been forgotten by some modern authors (e.g., Messiah, 1964, p. 140; Matthews, 1968, p. 27).

The former contention is refuted by an analysis of a measurement of a microscopic object by a macroscopic apparatus (Sec. 4). It follows from very simple con-

siderations that the final state of the system (object plus apparatus) *must*, in general, be a coherent superposition of macroscopically distinct "pointer positions." If the state vector is assumed to *completely* describe the individual system, i.e., if one assumes that the system simply does not have a value for any observable except those for which the state is an eigenvector (or perhaps those whose dispersion is very small), then one will be forced to the absurd conclusion that the "pointer" of the apparatus (a *macroscopic* object) has no definite position. The supposed *reduction of the state vector*, and all the difficulties and complications associated with it, are only artifacts of the vain attempt to retain the above assumption. But to what purpose? Under the more modest assumption that a state vector represents an *ensemble* of similarly prepared systems, the measurement process poses no particular problem.

The criticisms above also apply to certain hidden-variable theories in which a ψ field is associated with each individual system (Bohm, 1952; Bohm and Bub, 1966). Like Von Neumann's theory, these theories must also invoke a reduction of the ψ field upon measurement, which in the example of the correlated but spacially separated systems considered by EPR, would necessitate a signal passing instantaneously across the macroscopic distances between the systems. Since, as we have seen, such a drastic assumption is *not necessary* for a satisfactory interpretation of quantum theory, we see no reason for considering it further.

The *Uncertainty Principle* (Sec. 3) finds its natural interpretation as a lower bound on the statistical dispersion among similarily prepared system (this interpretation being *deduced*, not introduced *ad hoc*), and is not in any real sense related to the possible disturbance of a system by a measurement. The distinction between *measurement* and *state preparation* is essential for clarity. It is possible to extend the formalism of quantum theory by the introduction of *joint probability distributions* for position and momentum (Sec. 5). This demonstrates that there is no conflict with quantum theory in thinking of a particle as having definite (but, in general, unknown) values of both position and momentum, contrary to an earlier interpretation of the uncertainty principle.

The Statistical Interpretation does not prejudice the possibility of introducing *hidden variables* which would determine (in principle) the outcome of each individual measurement (Sec. 6). Although such models exist, Bell (1964) showed that they cannot, in general, satisfy the requirement that measurements on spacially separated noninteracting objects should be independent. It is ironic that this requirement which Einstein, Podolsky, and Rosen first used to refute the belief that a wave function could completely describe an individual system (a belief which stood opposed to hidden variables), should also prove disasterous to at least the simplest idea of hidden variables.

Many of the ideas expounded in this paper were expressed, in essence, by Einstein (1949) in his *Reply*

to Criticisms. In support of his thesis that a wave function must be considered to represent an ensemble of systems, and cannot reasonably be assumed to provide a complete description of an individual system, he considers the decay of a radioactive atom, with an automatic recording detector to register the decay time.[26] This example is a prototype of the one used in the Theory of Measurement, and it embodies all the essential physical content of the more general argument. In this *Reply* he also restated the conclusion of the EPR argument in the form which we have called the Theorem of EPR, and gave a brief discussion of the nontrivial aspect of the classical limit of the quantal state description.

A serious reading of Einstein's *Reply* should clear up any misconceptions to the effect that he rejected quantum theory or misunderstood its foundations. In fact, he understood the essentially *statistical* nature of quantum theory as well as any of his contemporaries, and better than many. His only objection was against the assumption that a wave function or state vector could exhaustively describe an *individual* system, which we have seen to be an unwarranted and troublesome assumption. This fact, and the fact that Einstein advocated a fully viable interpretation of quantum theory (essentially the Statistical Interpretation of this paper although he expressed himself more briefly), do not seem to have been appreciated by his critics.[27]

The foundations of quantum theory are subject to continuous discussion, and two almost coincident papers in the *American Journal of Physics*, taking nearly opposite points of view, deserve comment. Hartle (1968) has made a novel attempt to derive the statistical assertions of quantum theory from a quantal description of individual systems. The conceptual basis of this attempt is questionable, for, as Hartle admits, a quantum state is *not* an objective property of an individual system. That is to say, in the words of Blokhintsev (1968, p. 50),

"If [the wave function were a characteristic of a single particle] it would be of interest to perform such a measurement on a single particle (say an electron) which would allow us to determine its own individual wave function. No such measurement is possible."

But whereas Blokhintsev (and the present author) regards the quantum state as describing an *ensemble* of similarily prepared systems, Hartle considers it to describe the information possessed by some observer. The irrelevance of that interpretation has already been commented upon (Sec. 4.4).

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[26] Messiah's (1964, p. 158) reply to this argument is essentially that there are experimental arrangements from which it is impossible to determine a definite decay time. (His wording to the effect that they are *incompatible* with the *existence* of a time of decay is an unjustified overstatement). But this is irrelevant. The point is that it is *sometimes* (i.e., with suitable apparatus) possible to measure the decay time of a single atom, but this decay time is *never* predicted by quantum theory.

[27] See the articles by Pauli, Born, Heitler, Bohr, and Margenau in the volume edited by Schilpp (1949). However, Margenau's recent papers are more compatible with Einstein's views.

On the other hand Park (1968) has carefully studied the relations between *ensembles* and *individual* systems in statistical theories. He shows how one may proceed logically from the description of ensembles to individual systems in classical statistical mechanics, and how the attempt to proceed similarly in quantum statistical mechanics to an identification of a *pure state* operator with an individual system fails. The present paper is consistent with his conclusion; moreover we have shown that such an identification would lead to serious difficulties in principle.

Although this article has been concerned primarily with a consistent interpretation of quantum theory as it is presently formulated, one is naturally led to ask for hints of future developments. Except for eliminating some false starts, such as theories of wave function reduction, such hints are hard to find. Recognition that quantum states should refer to ensembles of similarily prepared systems would seem to open the door for hidden variables to control individual events. On the other hand, Bell's theorem seems to present a severe obstacle for any hidden-variable theory which reproduces exactly *all* of the predictions of quantum theory, but this need not be so for a theory which departs from the formalism of quantum theory and recovers it only in some limiting case. Presumably the next step must be a bold departure from the familiar formalism, as Einstein's theory of gravitation departed from that of Newton.

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Charges and Generators of Symmetry Transformations in Quantum Field Theory[*]

CLAUDIO A. ORZALESI[†]

Department of Physics, Columbia University, New York, New York

Within the Wightman approach to quantum field theory, we review and clarify the properties of formal charges, defined as space integrals for the fourth component of a local current. The conditions for a formal charge to determine an operator (generator) are discussed, in connection with the well-known theorems of Goldstone and of Coleman. The symmetry transformations generated by this operator—given its existence—are also studied in some detail. For generators in a scattering theory, we prove their additivity and thus provide a simple way to characterize them from their matrix elements between one-particle states. This characterization allows an immediate construction of the unitary operators implementing the symmetry transformations, and implies that all internal symmetry groups are necessarily compact. We also indicate how to construct interacting fields having definite internal quantum numbers. The present status of the proof of Noether's theorem and of its converse is discussed in the light of the rather delicate properties of formal charges.

CONTENTS

1. INTRODUCTION

Following the results of Goldstone *et al.* (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962) and of Coleman (1966), in recent years there has been a continual interest in the properties of formal charges. A formal charge Q is defined here as the space integral

of the zeroth component of a local four-vector current:

$$Q(x_0) = \int d\mathbf{x}\, j_0(x). \tag{1.1}$$

Quantities of this kind appear in the discussion of symmetries and broken symmetries in quantum field theory, and are one of the basic tools in the modern "current-algebraic" approach to elementary particle physics (Gell-Mann, 1962; Adler and Dashen, 1968).

It has been repeatedly emphasized (Kastler, Robinson, and Swieca, 1966; Schroer and Stichel, 1966; Dell'Antonio, 1967; Swieca, 1966; Katz, 1966; Fabri and Picasso, 1966; Fabri, Picasso, and Strocchi, 1967; and De Mottoni, 1967) that equations of the type (1.1) have rather delicate convergence properties, and that a certain care has to be exercised when considering such expressions. This fact limits the extent to which Q can be thought of as a generator of symmetry or broken symmetry transformations.[1] The same convergence properties are at the basis of Goldstone's theorem (Goldstone, 1961; Goldstone, Salam and Weinberg, 1962; Kastler, Robinson and Swieca, 1966), and of Coleman's theorems (Coleman, 1965 and 1966; Pohlmeyer, 1966; Schroer and Stichel, 1966; and

[1] The nomenclature as well as the mentioned restrictions will be clarified later on. For present purposes, a generator of symmetry transformations is to be identified with a self-adjoint operator which commutes with **P**, the momentum operator, and commutes with the S matrix. A conserved current leads to an exact symmetry if the associated charge is a generator of symmetry transformations. Spontaneously broken symmetries occur when current conservation does *not* imply the existence of a symmetry. Intrinsically broken symmetries arise when the current is not conserved.