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## A FUNCTIONAL CALCULUS OF FIRST ORDER BASED ON STRICT IMPLICATION

RUTH C. BARCAN

The following system<sup>1</sup> is an extension of the Lewis calculus S2 to include quantification.<sup>2</sup>

### Primitive symbols.

( ) {parentheses}.

$\sim$  {negation}.

$\diamond$  {modal operator}.

$\cdot$  {conjunction}.

( $\exists$ ) {existential quantifier, the blank space replaced by an appropriate variable}.

Propositional variables  $p, q, r, s, t, p_1, q_1, r_1, s_1, t_1, \dots, p_n, q_n, r_n, s_n, t_n, \dots$ .

Functional variables of degree 1, 2,  $\dots, n, \dots$ : monadic functional variables  $F_1, G_1, H_1, F_1^1, \dots$ ; dyadic functional variables  $F_2, G_2, H_2, F_2^1, \dots$ ;  $n$ -adic functional variables  $F_n, G_n, H_n, F_n^1, \dots; \dots$ .

**Formula** is defined as any finite sequence of primitive symbols.

**Syntactic notation.** Greek letters will be used as variables in the syntax language. Upper case Greek letters A, B,  $\Gamma$ , E, H,  $A_1, B_1, \Gamma_1, E_1, H_1, \dots, A_n, B_n, \Gamma_n, E_n, H_n$  represent formulas. Lower case Greek letters  $\alpha, \beta, \gamma, \delta, \alpha_1, \beta_1, \gamma_1, \delta_1, \dots, \alpha_n, \beta_n, \gamma_n, \delta_n$  represent individual variables unless otherwise specified.

**Well-formed formula.** Write "wff" for "well-formed formula" and "wf" for "well-formed." A wff is defined recursively as follows;

A propositional variable is wf.

$\beta(\alpha_1, \alpha_2, \dots, \alpha_n)$  is wf where  $\beta$  is an  $n$ -adic functional variable and  $\alpha_1, \alpha_2, \dots, \alpha_n$  are individual variables.

If A and B are wff's then  $\sim A, \diamond A, (A \cdot B), (\exists \alpha)A$  are wff's.

The only wff's are those which follow from this definition.

Capital Greek letters will hereafter be restricted to the representation of wff's.

**Bound and free individual variables.** An occurrence of an individual variable  $\alpha$  in a wff A is a bound occurrence if it is in a wf part of A of the form  $(\exists \alpha)B$ . Otherwise it is a free occurrence.

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<sup>1</sup> This paper is an extract from a dissertation being written in partial fulfillment of the requirements for the Ph.D. degree in Philosophy at Yale University.

The method for constructing this system was suggested in part by Alonzo Church's *Introduction to mathematical logic*, Princeton University Press, 1944.

<sup>2</sup> Lewis and Langford, *Symbolic logic*, The Century Co., 1932.

**Definition schemata.**  $=_{df}$  between two expressions indicates that the formula on the left abbreviates the formula on the right.

- $$\begin{aligned}
 (\alpha)A &=_{df} \sim(\exists\alpha)\sim A. \\
 (A \vee B) &=_{df} \sim(\sim A \cdot \sim B). \\
 (A \rightarrow B) &=_{df} \sim\Diamond(A \cdot \sim B). \\
 (A \equiv B) &=_{df} ((A \rightarrow B) \cdot (B \rightarrow A)). \\
 (A \supset B) &=_{df} \sim(A \cdot \sim B). \\
 (A \equiv B) &=_{df} ((A \supset B) \cdot (B \supset A)).
 \end{aligned}$$

The outermost parentheses will be omitted. The dot for conjunction will also be omitted wherever unambiguous. Rules will be distinguished by Roman numerals.

**Axiom schemata.**

1.  $(AB) \rightarrow (BA)$ .
2.  $(AB) \rightarrow A$ .
3.  $A \rightarrow (AA)$ .
4.  $(AB)\Gamma \rightarrow A(B\Gamma)$ .
5.  $((A \rightarrow B)(B \rightarrow \Gamma)) \rightarrow (A \rightarrow \Gamma)$ .
6.  $(A(A \rightarrow B)) \rightarrow B$ .
7.  $\Diamond(AB) \rightarrow \Diamond A$ .
8.  $(\alpha)A \rightarrow B$ , where  $\alpha$  and  $\beta$  are individual variables, no free occurrence of  $\alpha$  in  $A$  is in a wf'd part of  $A$  of the form  $(\beta)\Gamma$  and  $B$  results from the substitution of  $\beta$  for all free occurrences of  $\alpha$  in  $A$ .
9.  $(\alpha)(A \supset B) \rightarrow ((\alpha)A \supset (\alpha)B)$ .
10.  $A \rightarrow (\alpha)A$ , where  $\alpha$  is not free in  $A$ .
11.  $\Diamond(\exists\alpha)A \rightarrow (\exists\alpha)\Diamond A$ .

**Rules of inference.**

- I. From  $A$  and  $A \rightarrow B$  infer  $B$  (modus ponens, abbreviate: mod pon).
- II. From  $A$  and  $B$  infer  $AB$  (adjunction, abbreviate: adj).
- III. If  $A$ ,  $B$ , and  $\Gamma$  are such that  $E$  results from  $\Gamma$  by the substitution of  $B$  for one or more occurrences of  $A$  in  $\Gamma$ , and if  $A \equiv B$ , then infer  $\Gamma$  from  $E$  and  $E$  from  $\Gamma$  (substitution, abbreviate: subst).
- IV. If  $B$  is the result of substituting the individual variable  $\beta$  for all free occurrences of  $\alpha$  in  $A$  then infer  $(\beta)B$  from  $A$ . (generalization, abbreviate: gen)

**Proof** of a wff  $A$  is a finite list of wff's  $A_1, A_2, \dots, A_n$  such that each  $A_i$  is an axiom or derivable from the preceding formulas of the list by applying any of the rules of inference.  $A$  is provable if it is the  $A_n$  of such a list.

" $A$  is provable" will be abbreviated:  $\vdash A$ .

**Correspondence with Lewis system S2.** Axiom schemata 1–7 correspond to B1–B4, B6–B8.<sup>3</sup> B5 is derivable as shown by McKinsey.<sup>4</sup> Rules I, II, and

<sup>3</sup> See Lewis and Langford, p. 493.

<sup>4</sup> J. C. C. McKinsey, *Bulletin of the American Mathematical Society*, vol. 40 (1934), pp. 425–427.

III correspond to Lewis's rules of inference, adjunction and substitution (a).<sup>5</sup> We can dispense with substitution (b) since we have an infinite list of axioms.

The following notational changes will be assumed as made when referring to the theorems of *Symbolic logic*: Lewis's propositional variables replaced by capital Greek letters, and = replaced by  $\equiv$  or  $=_{df}$  according as an equivalence or a defining relation is indicated, and dots for brackets and brackets replaced by parentheses.

When reference is made in the proofs to a theorem in *Symbolic logic* the number will be italicized. Otherwise the numbers refer to the list of axioms and theorems developed here.

Some rules useful for facilitating proofs.

V. If  $\vdash A \rightarrow B$  then  $\vdash A \equiv (AB)$ .  
 $\vdash (A \rightarrow B) \equiv (A \rightarrow (AB))$  16.33  
 $\vdash A \rightarrow B$  hyp  
 $\vdash A \rightarrow (AB)$  subst  
 $\vdash (AB) \rightarrow A$  2  
 $\vdash (A \rightarrow (AB))((AB) \rightarrow A)$  adj  
 $\vdash A \equiv (AB)$  def

VI. If  $\vdash A \rightarrow B$  then  $\vdash \diamond A \rightarrow \diamond B$ .  
 $\vdash A \rightarrow B$  hyp  
 $\vdash A \equiv (AB)$  V  
 $\vdash \diamond(AB) \rightarrow \diamond B$  19.13  
 $\vdash \diamond A \rightarrow \diamond B$  subst

Assume that each step in the following proofs is preceded by  $\vdash$ .

VII. If  $\vdash A \rightarrow B$  then  $\vdash \sim \diamond \sim A \rightarrow \sim \diamond \sim B$ .  
 $A \rightarrow B$  hyp  
 $A \equiv (AB)$  V  
 $\sim \diamond \sim (AB) \rightarrow \sim \diamond \sim B$  19.24  
 $\sim \diamond \sim A \rightarrow \sim \diamond \sim B$  subst

VIII. If  $\vdash A_1 \rightarrow A_2, \vdash A_2 \rightarrow A_3, \dots, \vdash A_{n-1} \rightarrow A_n$  then  $\vdash A_1 \rightarrow A_n$ .  
 $(A_1 \rightarrow A_2)(A_2 \rightarrow A_3)$  hyp, adj  
 $((A_1 \rightarrow A_2)(A_2 \rightarrow A_3)) \rightarrow (A_1 \rightarrow A_3)$  5  
 $A_1 \rightarrow A_3$  mod pon  
 $(A_1 \rightarrow A_3)(A_3 \rightarrow A_4)$  hyp, adj  
 $((A_1 \rightarrow A_3)(A_3 \rightarrow A_4)) \rightarrow (A_1 \rightarrow A_4)$  5  
 $A_1 \rightarrow A_4$  mod pon  
 $\vdots$   
 $A_1 \rightarrow A_n$  mod pon

<sup>5</sup> See Lewis and Langford, p. 125.

- IX. If  $\vdash A \rightarrow B$  and  $\vdash \Gamma \rightarrow E$  then  $\vdash (A\Gamma) \rightarrow (BE)$ .  
 $((A \rightarrow B)(\Gamma \rightarrow E)) \rightarrow ((A\Gamma) \rightarrow (BE))$  19.68  
 $(A \rightarrow B)(\Gamma \rightarrow E)$  hyp, adj  
 $(A\Gamma) \rightarrow (BE)$  mod pon
- X. If  $\vdash A \rightarrow B$  then  $\vdash \sim B \rightarrow \sim A$ .  
 $A \rightarrow B$  hyp  
 $(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$  12.43  
 $\sim B \rightarrow \sim A$  mod pon
- XI. If  $\vdash A \equiv B$  then  $\vdash \sim A \equiv \sim B$ .  
 $((A \rightarrow B)(B \rightarrow A)) \rightarrow ((\sim A \rightarrow \sim B)(\sim B \rightarrow \sim A))$  12.1  
 $((A \rightarrow B)(B \rightarrow A)) \rightarrow ((\sim B \rightarrow \sim A)(\sim A \rightarrow \sim B))$  12.44, subst  
 $(A \rightarrow B)(B \rightarrow A)$  hyp, def  
 $(\sim B \rightarrow \sim A)(\sim A \rightarrow \sim B)$  mod pon  
 $\sim A \equiv \sim B$  12.15, subst, def
- XII. If  $\vdash A \rightarrow B$  and  $\vdash A \rightarrow \Gamma$  then  $\vdash A \rightarrow (B\Gamma)$ .  
 $((A \rightarrow B)(A \rightarrow \Gamma)) \rightarrow (A \rightarrow (B\Gamma))$  19.61  
 $(A \rightarrow B)(A \rightarrow \Gamma)$  hyp, adj  
 $(A \rightarrow (B\Gamma))$  mod pon
13.  $\vdash (\exists\alpha)A \equiv \sim(\alpha)\sim A$ .  
 $\sim\sim(\exists\alpha)A \equiv \sim(\alpha)\sim A$  12.1, def, 12.3, subst, XI  
 $(\exists\alpha)A \equiv \sim(\alpha)\sim A$  12.3, subst
14.  $\vdash (\exists\alpha)\sim A \equiv \sim(\alpha)A$ .  
 $(\exists\alpha)\sim A \equiv \sim(\alpha)\sim\sim A$  13  
 $(\exists\alpha)\sim A \equiv \sim(\alpha)A$  12.3, subst
15.  $\vdash \sim(\exists\alpha)A \equiv (\alpha)\sim A$ .  
 $(\exists\alpha)A \equiv \sim(\alpha)\sim A$  13  
 $\sim(\exists\alpha)A \equiv \sim\sim(\alpha)\sim A$  XI  
 $\sim(\exists\alpha)A \equiv (\alpha)\sim A$  12.3, subst
16.  $\vdash B \rightarrow (\exists\alpha)A$ , where  $B$ ,  $A$ , and  $\alpha$  are as in 8.  
 $(\alpha)\sim A \rightarrow \sim B$  8  
 $\sim\sim B \rightarrow \sim(\alpha)\sim A$  X  
 $B \rightarrow \sim(\alpha)\sim A$  12.3, subst  
 $B \rightarrow (\exists\alpha)A$  13, subst
17.  $\vdash (\alpha)A \rightarrow (\exists\alpha)A$ .  
 $(\alpha)A \rightarrow A$  8  
 $A \rightarrow (\exists\alpha)A$  16  
 $(\alpha)A \rightarrow (\exists\alpha)A$  V'''

18.  $\vdash (\alpha) \sim \diamond \sim A \rightarrow \sim \diamond \sim (\alpha) A.$   
 $\diamond (\exists \alpha) \sim A \rightarrow (\exists \alpha) \diamond \sim A$  11  
 $\sim (\exists \alpha) \diamond \sim A \rightarrow \sim \diamond (\exists \alpha) \sim A$  X  
 $(\alpha) \sim \diamond \sim A \rightarrow \sim \diamond (\exists \alpha) \sim A$  15, subst  
 $(\alpha) \sim \diamond \sim A \rightarrow \sim \diamond \sim (\alpha) \sim \sim A$  13, subst  
 $(\alpha) \sim \diamond \sim A \rightarrow \sim \diamond \sim (\alpha) A$  12.3, subst
19.  $\vdash (\alpha) (A \rightarrow B) \rightarrow ((\alpha) A \rightarrow (\alpha) B).$   
 $(\alpha) (A \supset B) \rightarrow ((\alpha) A \supset (\alpha) B)$  9  
 $\sim \diamond \sim (\alpha) (A \supset B) \rightarrow \sim \diamond \sim ((\alpha) A \supset (\alpha) B)$  VII  
 $(\alpha) \sim \diamond \sim (A \supset B) \rightarrow \sim \diamond \sim (\alpha) (A \supset B)$  18  
 $(\alpha) \sim \diamond \sim (A \supset B) \rightarrow \sim \diamond \sim ((\alpha) A \supset (\alpha) B)$  VIII  
 $(\alpha) (A \rightarrow B) \rightarrow ((\alpha) A \rightarrow (\alpha) B)$  18.7, subst
20.  $\vdash_{\sim}^{\square} A \equiv (\alpha) A$ , where  $\alpha$  is not free in  $A$ .  
 $A \rightarrow (\alpha) A$  10  
 $(\alpha) A \rightarrow A$  8  
 $(A \rightarrow (\alpha) A) ((\alpha) A \rightarrow A)$  adj  
 $A \equiv (\alpha) A$  def
21.  $\vdash A \equiv (\exists \alpha) A$ , where  $\alpha$  is not free in  $A$ .  
 $\sim A \equiv (\alpha) \sim A$  20, hyp  
 $\sim \sim A \equiv \sim (\alpha) \sim A$  XI  
 $A \equiv (\exists \alpha) A$  12.3, subst, 13
22.  $\vdash (\alpha) (A \rightarrow B) \rightarrow (A \rightarrow (\alpha) B)$ , where  $\alpha$  is not free in  $A$ .  
 $(\alpha) (A \rightarrow B) \rightarrow ((\alpha) A \rightarrow (\alpha) B)$  19  
 $(\alpha) (A \rightarrow B) \rightarrow (A \rightarrow (\alpha) B)$  20, subst
23.  $\vdash (\alpha) (A \rightarrow B) \rightarrow ((\exists \alpha) A \rightarrow B)$ , where  $\alpha$  is not free in  $B$ .  
 $(\alpha) (A \rightarrow B) \rightarrow (\alpha) (A \rightarrow B)$  12.1  
 $(\alpha) (A \rightarrow B) \rightarrow (\alpha) (\sim B \rightarrow \sim A)$  12.44, subst  
 $(\alpha) (\sim B \rightarrow \sim A) \rightarrow (\sim B \rightarrow (\alpha) \sim A)$  22  
 $(\alpha) (A \rightarrow B) \rightarrow (\sim B \rightarrow (\alpha) \sim A)$  VIII  
 $(\alpha) (A \rightarrow B) \rightarrow (\sim (\alpha) \sim A \rightarrow \sim \sim B)$  12.44, subst  
 $(\alpha) (A \rightarrow B) \rightarrow ((\exists \alpha) A \rightarrow B)$  13, 12.3, subst
- XIII. If  $\vdash A \rightarrow B$  then  $\vdash (\alpha) A \rightarrow (\alpha) B$ .  
 $A \rightarrow B$  hyp  
 $(\alpha) (A \rightarrow B)$  gen  
 $(\alpha) (A \rightarrow B) \rightarrow ((\alpha) A \rightarrow (\alpha) B)$  19  
 $(\alpha) A \rightarrow (\alpha) B$  mod pon
24.  $\vdash (\alpha) (A \equiv B) \rightarrow (\alpha) A \equiv (\alpha) B.$   
 $((A \rightarrow B) (B \rightarrow A)) \rightarrow (A \rightarrow B)$  2  
 $((A \rightarrow B) (B \rightarrow A)) \rightarrow (B \rightarrow A)$  12.17  
 $(\alpha) ((A \rightarrow B) (B \rightarrow A)) \rightarrow (\alpha) (A \rightarrow B)$  XIII  
 $(\alpha) ((A \rightarrow B) (B \rightarrow A)) \rightarrow (\alpha) (B \rightarrow A)$  XIII

$$\begin{array}{l}
(\alpha)(A \rightarrow B) \rightarrow ((\alpha)A \rightarrow (\alpha)B) \quad 19 \\
(\alpha)(B \rightarrow A) \rightarrow ((\alpha)B \rightarrow (\alpha)A) \quad 19 \\
(\alpha)((A \rightarrow B)(B \rightarrow A)) \rightarrow ((\alpha)A \rightarrow (\alpha)B) \quad \text{VIII} \\
(\alpha)((A \rightarrow B)(B \rightarrow A)) \rightarrow ((\alpha)B \rightarrow (\alpha)A) \quad \text{VIII} \\
(\alpha)((A \rightarrow B)(B \rightarrow A)) \rightarrow (((\alpha)A \rightarrow (\alpha)B)((\alpha)B \rightarrow (\alpha)A)) \quad \text{XII} \\
(\alpha)(A \equiv B) \rightarrow ((\alpha)A \equiv (\alpha)B) \quad \text{def}
\end{array}$$

XIV. If  $\vdash A \equiv B$  then  $\vdash (\alpha)A \equiv (\alpha)B$ .

$$\begin{array}{l}
A \equiv B \quad \text{hyp} \\
(\alpha)A \equiv (\alpha)A \quad 12.11 \\
(\alpha)A \equiv (\alpha)B \quad \text{subst}
\end{array}$$

25.  $\vdash (\alpha)(A \rightarrow B) \rightarrow ((\exists\alpha)A \rightarrow (\exists\alpha)B)$ .

$$\begin{array}{l}
(\alpha)(A \rightarrow B) \rightarrow (\alpha)(A \rightarrow B) \quad 12.1 \\
(\alpha)(A \rightarrow B) \rightarrow (\alpha)(\sim B \rightarrow \sim A) \quad 12.44, \text{subst} \\
(\alpha)(\sim B \rightarrow \sim A) \rightarrow ((\alpha)\sim B \rightarrow (\alpha)\sim A) \quad 19 \\
((\alpha)\sim B \rightarrow (\alpha)\sim A) \rightarrow (\sim(\alpha)\sim A \rightarrow \sim(\alpha)\sim B) \quad 12.43 \\
(\alpha)(A \rightarrow B) \rightarrow (\sim(\alpha)\sim A \rightarrow \sim(\alpha)\sim B) \quad \text{VIII} \\
(\alpha)(A \rightarrow B) \rightarrow ((\exists\alpha)A \rightarrow (\exists\alpha)B) \quad 13, \text{subst}
\end{array}$$

XV. If  $\vdash A \rightarrow B$  then  $\vdash (\exists\alpha)A \rightarrow (\exists\alpha)B$ .

Like XIII, using 25 in place of 19.

26.  $\vdash (\alpha)(AB) \rightarrow ((\alpha)A (\alpha)B)$ .

$$\begin{array}{l}
(AB) \rightarrow A \quad 2 \\
(\alpha)(AB) \rightarrow (\alpha)A \quad \text{XIII} \\
(AB) \rightarrow B \quad 12.17 \\
(\alpha)(AB) \rightarrow (\alpha)B \quad \text{XIII} \\
(\alpha)(AB) \rightarrow ((\alpha)A (\alpha)B) \quad \text{XII}
\end{array}$$

27.  $\vdash (\exists\alpha)(AB) \rightarrow ((\exists\alpha)A (\exists\alpha)B)$ .

Like 26 using XV in place of XIII.

28.  $\vdash ((\alpha)A (\alpha)B) \rightarrow (\alpha)(AB)$ .

$$\begin{array}{l}
(\alpha)A \rightarrow A \quad 8 \\
(\alpha)B \rightarrow B \quad 8 \\
((\alpha)A (\alpha)B) \rightarrow (AB) \quad \text{IX} \\
(\alpha)(((\alpha)A (\alpha)B) \rightarrow (AB)) \rightarrow (((\alpha)A (\alpha)B) \rightarrow (\alpha)(AB)) \quad 22 \\
(\alpha)(((\alpha)A (\alpha)B) \rightarrow (AB)) \quad \text{gen} \\
((\alpha)A (\alpha)B) \rightarrow (\alpha)(AB) \quad \text{mod pon}
\end{array}$$

29.  $\vdash ((\alpha)A (\alpha)B) \equiv (\alpha)(AB)$ .

$$\begin{array}{l}
(((\alpha)A (\alpha)B) \rightarrow (\alpha)(AB))((\alpha)(AB) \rightarrow ((\alpha)A (\alpha)B)) \quad 28, 26, \text{adj} \\
((\alpha)A (\alpha)B) \equiv (\alpha)(AB) \quad \text{def}
\end{array}$$

30.  $\vdash (\exists\alpha)A \rightarrow (\exists\alpha)(A \vee B)$ .

$$\begin{array}{l}
A \rightarrow (A \vee B) \quad 13.2 \\
(\exists\alpha)A \rightarrow (\exists\alpha)(A \vee B) \quad \text{XV}
\end{array}$$

31.  $\vdash (\exists\alpha)B \rightarrow (\exists\alpha)(A \vee B)$ .  
Like 30 using 13.21 in place of 13.2.
32.  $\vdash ((\exists\alpha)A \vee (\exists\alpha)B) \rightarrow (\exists\alpha)(A \vee B)$ .  
 $((\exists\alpha)A \rightarrow (\exists\alpha)(A \vee B))((\exists\alpha)B \rightarrow (\exists\alpha)(A \vee B)) \rightarrow$   
 $((\exists\alpha)A \vee (\exists\alpha)B) \rightarrow (\exists\alpha)(A \vee B)$  19.65  
 $((\exists\alpha)A \rightarrow (\exists\alpha)(A \vee B))((\exists\alpha)B \rightarrow (\exists\alpha)(A \vee B))$  30, 31, adj  
 $((\exists\alpha)A \vee (\exists\alpha)B) \rightarrow (\exists\alpha)(A \vee B)$  mod pon
33.  $\vdash (\exists\alpha)(A \vee B) \equiv ((\exists\alpha)A \vee (\exists\alpha)B)$ .  
 $((\alpha)\sim A (\alpha)\sim B) \equiv (\alpha)(\sim A \sim B)$  29  
 $\sim(\sim(\alpha)\sim A \vee \sim(\alpha)\sim B) \equiv (\alpha)\sim(\sim\sim A \vee \sim\sim B)$   
14.21, subst, def  
 $\sim\sim(\sim(\alpha)\sim A \vee \sim(\alpha)\sim B) \equiv \sim(\alpha)\sim(\sim\sim A \vee \sim\sim B)$  XI  
 $(\sim(\alpha)\sim A \vee \sim(\alpha)\sim B) \equiv \sim(\alpha)\sim(A \vee B)$  12.3, subst  
 $((\exists\alpha)A \vee (\exists\alpha)B) \equiv (\exists\alpha)(A \vee B)$  13, subst
34.  $\vdash (\alpha)A \rightarrow (\alpha)(A \vee B)$ .  
Like 30 using XIII instead of XV.
35.  $\vdash (\alpha)B \rightarrow (\alpha)(A \vee B)$ .  
Like 31 using XIII in place of XV.
36.  $\vdash ((\alpha)A \vee (\alpha)B) \rightarrow (\alpha)(A \vee B)$ .  
Like 32 using 34 and 35 in place of 30 and 31.

**Extension of the rule of substitution.**

XVI. If the wff's  $\Gamma$ ,  $E$ , and  $B$  are such that  $B$  results from  $A$  by the substitution of  $E$  for one or more occurrences of  $\Gamma$  in  $A$  and if  $(\alpha_1)(\alpha_2) \dots (\alpha_n)(\Gamma \equiv E)$  where  $\alpha_1, \alpha_2, \dots, \alpha_n$  is a complete list of the free variables in  $\Gamma$  and  $E$  then we may infer  $B$  from  $A$  and  $A$  from  $B$ .

$$(\alpha_1)(\alpha_2) \dots (\alpha_n)(\Gamma \equiv E) \rightarrow (\Gamma \equiv E)$$

successive applications of 8

$$(\alpha_1)(\alpha_2) \dots (\alpha_n)(\Gamma \equiv E) \quad \text{hyp}$$

$$\Gamma \equiv E \quad \text{mod pon}$$

XVI follows from the rule of substitution.

XVII. If  $\vdash A \rightarrow B$  and  $\alpha$  does not occur free in  $A$  then  $\vdash A \rightarrow (\alpha)B$ .  
 $A \rightarrow B$  hyp  
 $(\alpha)(A \rightarrow B)$  gen  
 $(\alpha)(A \rightarrow B) \rightarrow (A \rightarrow (\alpha)B)$ , where as above. 22  
 $A \rightarrow (\alpha)B$  mod pon

XVIII. If  $\vdash A \rightarrow B$  and  $\alpha$  does not occur free in  $B$  then  $\vdash (\exists\alpha)A \rightarrow B$ .  
Proof like XVII, using 23 instead of 22.

37.  $\vdash (\exists\alpha)\diamond A \rightarrow \diamond(\exists\alpha)A$ .  
 $A \rightarrow (\exists\alpha)A$  16  
 $\diamond A \rightarrow \diamond(\exists\alpha)A$  VI  
 $(\exists\alpha)\diamond A \rightarrow \diamond(\exists\alpha)A$  XVIII



38.  $\vdash (\exists\alpha)\diamond A \equiv \diamond(\exists\alpha)A.$   
 $((\exists\alpha)\diamond A \rightarrow \diamond(\exists\alpha)A)(\diamond(\exists\alpha)A \rightarrow (\exists\alpha)\diamond A)$  37, 11, adj  
 $(\exists\alpha)\diamond A \equiv \diamond(\exists\alpha)A$  def
39.  $\vdash (\alpha)\sim\diamond\sim A \equiv \sim\diamond\sim(\alpha)A.$   
 $(\exists\alpha)\diamond\sim A \equiv \diamond(\exists\alpha)\sim A$  38  
 $\sim(\alpha)\sim\diamond\sim A \equiv \diamond\sim(\alpha)\sim\sim A$  13, subst  
 $\sim\sim(\alpha)\sim\diamond\sim A \equiv \sim\diamond\sim(\alpha)\sim\sim A$  XI  
 $(\alpha)\sim\diamond\sim A \equiv \sim\diamond\sim(\alpha)A$  12.3, subst
40.  $\vdash \diamond(\alpha)A \rightarrow (\alpha)\diamond A.$   
 $(\alpha)A \rightarrow A$  8  
 $\diamond(\alpha)A \rightarrow \diamond A$  VI  
 $\diamond(\alpha)A \rightarrow (\alpha)\diamond A$  XVII
41.  $\vdash (\alpha)A \rightarrow (\beta)B$ , where no free occurrence of  $\alpha$  in  $A$  is in a well-formed part of  $A$  of the form  $(\beta)\Gamma$ ,  $B$  is formed from  $A$  by replacing all free occurrences of  $\alpha$  in  $A$  by  $\beta$ , and there is no free occurrence of  $\beta$  in  $A$ .  
 $(\alpha)A \rightarrow B$  8  
 $(\alpha)A \rightarrow (\beta)B$  XVII
42.  $\vdash (\alpha)A \equiv (\beta)B$ , where as in 41.  
 $(\alpha)A \rightarrow (\beta)B$  41  
 $(\beta)B \rightarrow (\alpha)A$  41  
 $(\alpha)A \equiv (\beta)B$  adj and def
43.  $\vdash (\exists\alpha)A \equiv (\exists\beta)B$ , where as in 41.  
 $(\alpha)\sim A \equiv (\beta)\sim B$  42  
 $\sim(\alpha)\sim A \equiv \sim(\beta)\sim B$  XI  
 $(\exists\alpha)A \equiv (\exists\beta)B$  13, subst
44.  $\vdash (\alpha)(A \equiv B) \rightarrow ((\exists\alpha)A \equiv (\exists\alpha)B).$   
 $(\alpha)(A \rightarrow B) \rightarrow ((\exists\alpha)A \rightarrow (\exists\alpha)B)$  25  
 $(\alpha)(B \rightarrow A) \rightarrow ((\exists\alpha)B \rightarrow (\exists\alpha)A)$  25  
 $((\alpha)(A \rightarrow B)(\alpha)(B \rightarrow A)) \rightarrow (((\exists\alpha)A \rightarrow (\exists\alpha)B)((\exists\alpha)B \rightarrow (\exists\alpha)A))$   
IX  
 $(\alpha)((A \rightarrow B)(B \rightarrow A)) \rightarrow (((\exists\alpha)A \rightarrow (\exists\alpha)B)((\exists\alpha)B \rightarrow (\exists\alpha)A))$   
29, subst  
 $(\alpha)(A \equiv B) \rightarrow ((\exists\alpha)A \equiv (\exists\alpha)B)$  def
45.  $\vdash (A \rightarrow (\alpha)B) \rightarrow (\alpha)(A \rightarrow B)$ , where  $\alpha$  is not free in  $A$ .  
 $(\exists\alpha)(A \sim B) \rightarrow ((\exists\alpha)A (\exists\alpha)\sim B)$  27  
 $(\exists\alpha)(A \sim B) \rightarrow (A (\exists\alpha)\sim B)$  21, subst  
 $(\exists\alpha)(A \sim B) \rightarrow (A \sim(\alpha)\sim\sim B)$  13, subst  
 $(\exists\alpha)(A \sim B) \rightarrow (A \sim(\alpha)B)$  12.3, subst  
 $\sim(A \sim(\alpha)B) \rightarrow \sim(\exists\alpha)(A \sim B)$  X  
 $(A \supset (\alpha)B) \rightarrow \sim(\exists\alpha)\sim(A \supset B)$  14.12, subst  
 $\sim\diamond\sim(A \supset (\alpha)B) \rightarrow \sim\diamond\sim(\alpha)(A \supset B)$  13, subst, VII  
 $\sim\diamond\sim(A \supset (\alpha)B) \rightarrow (\alpha)\sim\diamond\sim(A \supset B)$  39, subst  
 $(A \rightarrow (\alpha)B) \rightarrow (\alpha)(A \rightarrow B)$  18.7, subst

46.  $\vdash (A \rightarrow (\alpha)B) \equiv (\alpha)(A \rightarrow B)$ , where  $\alpha$  is not free in  $A$ .  
 $((A \rightarrow (\alpha)B) \rightarrow (\alpha)(A \rightarrow B))((\alpha)(A \rightarrow B) \rightarrow (A \rightarrow (\alpha)B))$ ,  
 where as above. 22, 45, adj  
 $(A \rightarrow (\alpha)B) \equiv (\alpha)(A \rightarrow B)$  def
47.  $\vdash (\exists\alpha)(B \rightarrow A) \rightarrow ((\alpha)B \rightarrow A)$ , where  $\alpha$  is not free in  $A$ .  
 $(\alpha)(B \sim A) \rightarrow (B \sim A)$  8  
 $((\alpha)B (\alpha)\sim A) \rightarrow (B \sim A)$  29, subst  
 $((\alpha)B \sim A) \rightarrow (B \sim A)$  20, subst  
 $\diamond((\alpha)B \sim A) \rightarrow \diamond(B \sim A)$  VI  
 $\sim\diamond(B \sim A) \rightarrow \sim\diamond((\alpha)B \sim A)$  X  
 $(\exists\alpha)\sim\diamond(B \sim A) \rightarrow \sim\diamond((\alpha)B \sim A)$  XVIII  
 $(\exists\alpha)(B \rightarrow A) \rightarrow ((\alpha)B \rightarrow A)$  def
48.  $\vdash (\exists\alpha)(A \rightarrow B) \rightarrow (A \rightarrow (\exists\alpha)B)$ , where  $\alpha$  is not free in  $A$ .  
 $B \rightarrow (\exists\alpha)B$  16  
 $\sim(\exists\alpha)B \rightarrow \sim B$  X  
 $(\sim(\exists\alpha)B \rightarrow \sim B) \rightarrow ((A \sim(\exists\alpha)B) \rightarrow (A \sim B))$   
 19.6, 12.15, subst  
 $(A \sim(\exists\alpha)B) \rightarrow (A \sim B)$  mod pon  
 $\diamond(A \sim(\exists\alpha)B) \rightarrow \diamond(A \sim B)$  VI  
 $\sim\diamond(A \sim B) \rightarrow \sim\diamond(A \sim(\exists\alpha)B)$  X  
 $(\exists\alpha)\sim\diamond(A \sim B) \rightarrow \sim\diamond(A \sim(\exists\alpha)B)$  XVIII  
 $(\exists\alpha)(A \rightarrow B) \rightarrow (A \rightarrow (\exists\alpha)B)$  def
- The above two theorems can be regarded as special cases of the more general principle  $(\exists\alpha)\sim\diamond\sim A \rightarrow \sim\diamond\sim(\exists\alpha)A$ .<sup>6</sup> Their converses are not provable.
49.  $\vdash (\exists\alpha)(A \rightarrow B) \rightarrow ((\alpha)A \rightarrow (\exists\alpha)B)$ .  
 $\sim(\exists\alpha)B \rightarrow \sim B$  step 2 of 48  
 $(\alpha)A \rightarrow A$  8  
 $((\alpha)A \sim(\exists\alpha)B) \rightarrow (A \sim B)$  IX  
 $\diamond((\alpha)A \sim(\exists\alpha)B) \rightarrow \diamond(A \sim B)$  VI  
 $\sim\diamond(A \sim B) \rightarrow \sim\diamond((\alpha)A \sim(\exists\alpha)B)$  XI  
 $(\exists\alpha)\sim\diamond(A \sim B) \rightarrow \sim\diamond((\alpha)A \sim(\exists\alpha)B)$  XVIII  
 $(\exists\alpha)(A \rightarrow B) \rightarrow ((\alpha)A \rightarrow (\exists\alpha)B)$  def
50.  $\vdash (\alpha)(B \rightarrow A) \rightarrow ((\alpha)B \rightarrow A)$ .  
 $(\alpha)B \rightarrow B$  8  
 $((\alpha)B \rightarrow B) \rightarrow (((\alpha)B \sim A) \rightarrow (B \sim A))$  19.6  
 $((\alpha)B \sim A) \rightarrow (B \sim A)$  mod pon  
 $\diamond((\alpha)B \sim A) \rightarrow \diamond(B \sim A)$  VI  
 $\sim\diamond(B \sim A) \rightarrow \sim\diamond((\alpha)B \sim A)$  XI  
 $(\alpha)\sim\diamond(B \sim A) \rightarrow (\alpha)\sim\diamond((\alpha)B \sim A)$  XIII  
 $(\alpha)\sim\diamond((\alpha)B \sim A) \rightarrow \sim\diamond((\alpha)B \sim A)$  8  
 $(\alpha)\sim\diamond(B \sim A) \rightarrow \sim\diamond((\alpha)B \sim A)$  VIII  
 $(\alpha)(B \rightarrow A) \rightarrow ((\alpha)B \rightarrow A)$  def

<sup>6</sup>  $(\exists\alpha)\sim\diamond\sim A \rightarrow \sim\diamond\sim(\exists\alpha)A$  follows easily from 40.

51.  $\vdash (\alpha)(A \vee B) \equiv (A \vee (\alpha)B)$ , where  $\alpha$  is not free in  $A$ .  
 $((\alpha)A \vee (\alpha)B) \rightarrow (\alpha)(A \vee B)$  36  
 $(A \vee (\alpha)B) \rightarrow (\alpha)(A \vee B)$ , where etc. 20, subst  
 Lemma:<sup>7</sup>  $\vdash (\alpha)(\Gamma \supset E) \rightarrow (\Gamma \supset (\alpha)E)$  where  $\alpha$  is not free in  $\Gamma$ .  
 $(\alpha)(\Gamma \supset E) \rightarrow ((\alpha)\Gamma \supset (\alpha)E)$  9  
 $(\alpha)(\Gamma \supset E) \rightarrow (\Gamma \supset (\alpha)E)$  20, subst.  
 $(\alpha)(\sim A \supset B) \rightarrow (\sim A \supset (\alpha)B)$  Lemma  
 $(\alpha)(A \vee B) \rightarrow (A \vee (\alpha)B)$  14.2, subst  
 $((\alpha)(A \vee B) \rightarrow (A \vee (\alpha)B))((A \vee (\alpha)B) \rightarrow (\alpha)(A \vee B))$  adj  
 $(\alpha)(A \vee B) \equiv (A \vee (\alpha)B)$  def
52.  $\vdash (\alpha)(A \vee B) \rightarrow ((\alpha)A \vee (\exists\alpha)B)$ .  
 Lemma:  $\vdash (\alpha)(A \supset B) \rightarrow ((\exists\alpha)A \supset (\exists\alpha)B)$ .  
 Like 25 using 9 instead of 19.  
 $(\alpha)(\sim A \supset B) \rightarrow ((\exists\alpha)\sim A \supset (\exists\alpha)B)$  Lemma  
 $(\alpha)(A \vee B) \rightarrow (\sim(\exists\alpha)\sim A \vee (\exists\alpha)B)$  14.12, 12.3, subst  
 $(\alpha)(A \vee B) \rightarrow ((\alpha)A \vee (\exists\alpha)B)$  def
53.  $\vdash (A (\exists\alpha)B) \equiv (\exists\alpha)(AB)$ , where  $\alpha$  is not free in  $A$ .  
 $(\alpha)(\sim A \vee \sim B) \equiv (\sim A \vee (\alpha)\sim B)$ , where etc. 51  
 $(\alpha)\sim(\sim\sim A \sim\sim B) \equiv \sim(\sim\sim A \sim(\alpha)\sim B)$  def  
 $(\alpha)\sim(AB) \equiv \sim(A \sim(\alpha)\sim B)$  12.3, subst  
 $\sim(\alpha)\sim(AB) \equiv \sim\sim(A \sim(\alpha)\sim B)$  XI  
 $(\exists\alpha)(AB) \equiv (A (\exists\alpha)B)$  12.3, subst, 13
54.  $\vdash (\alpha)\sim\Diamond A \rightarrow (\alpha)(A \rightarrow B)$ .  
 $\sim\Diamond A \rightarrow (A \rightarrow B)$  19.74  
 $(\alpha)\sim\Diamond A \rightarrow (\alpha)(A \rightarrow B)$  XIII
55.  $\vdash \sim\Diamond(\exists\alpha)A \rightarrow (\alpha)(A \rightarrow B)$ .  
 $(\alpha)\sim\Diamond A \rightarrow (\alpha)(A \rightarrow B)$  54  
 $\sim(\exists\alpha)\sim\sim\Diamond A \rightarrow (\alpha)(A \rightarrow B)$  def  
 $\sim(\exists\alpha)\Diamond A \rightarrow (\alpha)(A \rightarrow B)$  12.3, subst  
 $\sim\Diamond(\exists\alpha)A \rightarrow (\alpha)(A \rightarrow B)$  38, subst
56.  $\vdash \sim\Diamond\sim(\alpha)A \rightarrow (\alpha)(B \rightarrow A)$ .  
 $\sim\Diamond\sim A \rightarrow (B \rightarrow A)$  19.75  
 $(\alpha)\sim\Diamond\sim A \rightarrow (\alpha)(B \rightarrow A)$  XIII  
 $\sim\Diamond\sim(\alpha)A \rightarrow (\alpha)(B \rightarrow A)$  39, subst
57.  $\vdash \sim\Diamond(\exists\alpha)(AB) \rightarrow (\alpha)(A \rightarrow \sim B)$ .  
 $\sim\Diamond(\exists\alpha)(AB) \rightarrow \sim\Diamond(\exists\alpha)(AB)$  12.1  
 $\sim\Diamond(\exists\alpha)(AB) \rightarrow \sim(\exists\alpha)\Diamond(AB)$  38, subst  
 $\sim\Diamond(\exists\alpha)(AB) \rightarrow \sim(\exists\alpha)\sim(A \rightarrow \sim B)$  def, 12.3, subst  
 $\sim\Diamond(\exists\alpha)(AB) \rightarrow (\alpha)(A \rightarrow \sim B)$  def

<sup>7</sup> This Lemma might have been used as an axiom schema in place of 9 and 10.

58.  $\vdash (\alpha)(A \rightarrow (B \vee \Gamma)) \equiv \sim \diamond \sim (B \vee (\alpha)(A \supset \Gamma))$ , where  $\alpha$  is not free in  $B$ .

$$\begin{aligned} (\alpha) \sim \diamond (A \sim (B \vee \Gamma)) &\equiv (\alpha) \sim \diamond (A \sim (B \vee \Gamma)) && 12.11 \\ (\alpha) \sim \diamond (A \sim (B \vee \Gamma)) &\equiv (\alpha) \sim \diamond \sim (\sim A \vee \sim \sim (B \vee \Gamma)) && 14.21, \text{ subst} \\ (\alpha) \sim \diamond (A \sim (B \vee \Gamma)) &\equiv (\alpha) \sim \diamond \sim (\sim A \vee (B \vee \Gamma)) && 12.3, \text{ subst} \\ (\alpha) \sim \diamond (A \sim (B \vee \Gamma)) &\equiv (\alpha) \sim \diamond \sim (B \vee (\sim A \vee \Gamma)) && 13.41, \text{ subst} \\ (\alpha) \sim \diamond (A \sim (B \vee \Gamma)) &\equiv \sim \diamond \sim (\alpha)(B \vee (\sim A \vee \Gamma)) && 39, \text{ subst} \\ (\alpha) \sim \diamond (A \sim (B \vee \Gamma)) &\equiv \sim \diamond \sim (B \vee (\alpha)(\sim A \vee \Gamma)) && 51, \text{ subst} \\ (\alpha)(A \rightarrow (B \vee \Gamma)) &\equiv \sim \diamond \sim (B \vee (\alpha)(A \supset \Gamma)) && \text{def, 14.2, subst} \end{aligned}$$

59.  $\vdash ((\alpha)(A \rightarrow B) (\alpha)(B \rightarrow \Gamma)) \rightarrow (\alpha)(A \rightarrow \Gamma)$ .  
 $((A \rightarrow B)(B \rightarrow \Gamma)) \rightarrow (A \rightarrow \Gamma)$  5  
 $(\alpha)((A \rightarrow B)(B \rightarrow \Gamma)) \rightarrow (\alpha)(A \rightarrow \Gamma)$  XIII  
 $((\alpha)(A \rightarrow B) (\alpha)(B \rightarrow \Gamma)) \rightarrow (\alpha)(A \rightarrow \Gamma)$  29, subst

60.  $\vdash ((\alpha)(A \rightarrow B) \Gamma) \rightarrow E$ , where  $\Gamma$  is the result of replacing all free occurrences of  $\alpha$  in  $A$  by  $\beta$ , and  $\alpha$  does not occur in a well-formed part of  $A \rightarrow B$  of the form  $(\beta)H$ , and  $E$  is the result of replacing all free occurrences of  $\alpha$  in  $B$  by  $\beta$ .

Lemma: If  $\vdash A \rightarrow (B \rightarrow \Gamma)$  then  $\vdash (AB) \rightarrow \Gamma$ .  
 $(A \rightarrow (B \rightarrow \Gamma)) \rightarrow ((BA) \rightarrow (B(B \rightarrow \Gamma)))$

19.6, 12.15, subst

$$\begin{aligned} A \rightarrow (B \rightarrow \Gamma) &\quad \text{hyp} \\ (BA) \rightarrow (B(B \rightarrow \Gamma)) &\quad \text{mod pon} \\ (B(B \rightarrow \Gamma)) \rightarrow \Gamma &\quad 6 \\ (AB) \rightarrow \Gamma &\quad \text{VIII, 12.15, subst} \\ (\alpha)(A \rightarrow B) \rightarrow (\Gamma \rightarrow E), &\quad \text{where as above.} \quad 8 \\ ((\alpha)(A \rightarrow B)\Gamma) \rightarrow E &\quad \text{Lemma} \end{aligned}$$

59 and 60 are two instances of the syllogism in Barbara.

61.  $\vdash ((\alpha)(A \rightarrow B) (\alpha)(A \rightarrow \Gamma)) \equiv (\alpha)(A \rightarrow (B\Gamma))$ .  
 $((A \rightarrow B)(A \rightarrow \Gamma)) \rightarrow (A \rightarrow (B\Gamma))$  19.63  
 $(\alpha)((A \rightarrow B)(A \rightarrow \Gamma)) \rightarrow (\alpha)(A \rightarrow (B\Gamma))$  XIV  
 $((\alpha)(A \rightarrow B) (\alpha)(A \rightarrow \Gamma)) \rightarrow (\alpha)(A \rightarrow (B\Gamma))$  29, subst

62.  $\vdash ((\alpha)(A \rightarrow B) (\exists \alpha)A) \rightarrow (\exists \alpha)B$ .  
 $(\alpha)(A \rightarrow B) \rightarrow ((\exists \alpha)A \rightarrow (\exists \alpha)B)$  25  
 $((\alpha)(A \rightarrow B) (\exists \alpha)A) \rightarrow (\exists \alpha)B$  Lemma of 60

63.  $\vdash ((\alpha)(A \equiv B) (\alpha)(B \equiv \Gamma)) \rightarrow (\alpha)(A \equiv \Gamma)$ .  
 $((A \equiv B)(B \equiv \Gamma)) \rightarrow (A \equiv \Gamma)$  5, IX, def  
 Remainder of proof like 59.

64.  $\vdash (\alpha)(A \equiv B) \equiv (\alpha)(\sim B \equiv \sim A)$ .  
 Steps 1, 2 of XI, XIV, and def.

65.  $\vdash ((\alpha)(A \rightarrow B) \sim \diamond \sim (\alpha)A) \rightarrow \sim \diamond \sim (\alpha)B.$   
 $((A \rightarrow B) \sim \diamond \sim A) \rightarrow \sim \diamond \sim B$  18.53  
 $(\alpha)((A \rightarrow B) \sim \diamond \sim A) \rightarrow (\alpha)\sim \diamond \sim B$  XIII  
 $((\alpha)(A \rightarrow B) (\alpha)\sim \diamond \sim A) \rightarrow (\alpha)\sim \diamond \sim B$  29, subst  
 $((\alpha)(A \rightarrow B) \sim \diamond \sim (\alpha)A) \rightarrow \sim \diamond \sim (\alpha)B$  39, subst
66.  $\vdash ((\alpha)(A \rightarrow B) (\alpha)\diamond A) \rightarrow (\alpha)\diamond B.$   
 Like 65 using 18.51 instead of 18.53.
67.  $\vdash (\sim \diamond \sim (\alpha)A (\alpha)((AB) \rightarrow \Gamma)) \rightarrow (\alpha)(B \rightarrow \Gamma).$   
 $(\sim \diamond \sim A ((AB) \rightarrow \Gamma)) \rightarrow (B \rightarrow \Gamma)$  18.61  
 $(\alpha)(\sim \diamond \sim A ((AB) \rightarrow \Gamma)) \rightarrow (\alpha)(B \rightarrow \Gamma)$  XIII  
 $((\alpha)\sim \diamond \sim A (\alpha)((AB) \rightarrow \Gamma)) \rightarrow (\alpha)(B \rightarrow \Gamma)$  29, subst  
 $(\sim \diamond \sim (\alpha)A (\alpha)((AB) \rightarrow \Gamma)) \rightarrow (\alpha)(B \rightarrow \Gamma)$  39, subst
68.  $\vdash ((\exists \alpha)\diamond A \vee (\exists \alpha)\diamond B) \equiv (\exists \alpha)\diamond(A \vee B).$   
 $(\diamond A \vee \diamond B) \equiv \diamond(A \vee B)$  19.82  
 $(\exists \alpha)(\diamond A \vee \diamond B) \equiv (\exists \alpha)\diamond(A \vee B)$  Like proof of XIV.  
 $((\exists \alpha)\diamond A \vee (\exists \alpha)\diamond B) \equiv (\exists \alpha)\diamond(A \vee B)$  33, subst
69.  $\vdash (\diamond(\exists \alpha)A \vee \diamond(\exists \alpha)B) \equiv \diamond(\exists \alpha)(A \vee B).$  68, subst, 38
70.  $\vdash (\sim \diamond \sim (\alpha)A \sim \diamond \sim (\alpha)B) \rightarrow (\alpha)(A \equiv B).$   
 Like 65 using 19.84 instead of 18.53.
71.  $\vdash (\sim \diamond(\exists \alpha)A \sim \diamond(\exists \alpha)B) \rightarrow (\alpha)(A \equiv B).$   
 $(\sim \diamond A \sim \diamond B) \rightarrow (A \equiv B)$  19.83  
 $(\alpha)(\sim \diamond A \sim \diamond B) \rightarrow (\alpha)(A \equiv B)$  XIII  
 $((\alpha)\sim \diamond A (\alpha)\sim \diamond B) \rightarrow (\alpha)(A \equiv B)$  29, subst  
 $(\sim(\exists \alpha)\sim \sim \diamond A \sim(\exists \alpha)\sim \sim \diamond B) \rightarrow (\alpha)(A \equiv B)$  def  
 $(\sim(\exists \alpha)\diamond A \sim(\exists \alpha)\diamond B) \rightarrow (\alpha)(A \equiv B)$  12.3, subst  
 $(\sim \diamond(\exists \alpha)A \sim \diamond(\exists \alpha)B) \rightarrow (\alpha)(A \equiv B)$  38, subst

Definition:  $\Box A =_{df} \sim \diamond \sim A.$

72.  $\vdash \Box(\alpha_1)(\alpha_2) \cdots (\alpha_m)A \equiv (\alpha_1)(\alpha_2) \cdots (\alpha_m)\Box A.$   
 39, successive applications of subst.

The following theorems are not proved in *Symbolic logic*. Some of them will be useful in proving XIX below.

73.  $\vdash (A \rightarrow B) \rightarrow (\diamond A \supset \diamond B).$   
 $((A \rightarrow B) \diamond A) \rightarrow \diamond B) \rightarrow (((A \rightarrow B) \diamond A) \rightarrow \diamond B)$  12.1  
 $((A \rightarrow B) \diamond A) \rightarrow \diamond B) \rightarrow \sim \diamond(((A \rightarrow B) \diamond A) \sim \diamond B))$  def  
 $((A \rightarrow B) \diamond A) \rightarrow \diamond B) \rightarrow \sim \diamond((A \rightarrow B)(\diamond A \sim \diamond B))$   
 12.5, subst  
 $((A \rightarrow B) \diamond A) \rightarrow \diamond B) \rightarrow ((A \rightarrow B) \rightarrow \sim(\diamond A \sim \diamond B))$   
 def, 12.3, subst  
 $((A \rightarrow B) \diamond A) \rightarrow \diamond B) \rightarrow ((A \rightarrow B) \rightarrow (\diamond A \supset \diamond B))$  def  
 $((A \rightarrow B) \diamond A) \rightarrow \diamond B$  18.51  
 $(A \rightarrow B) \rightarrow (\diamond A \supset \diamond B)$  mod pon

74.  $\vdash (A \rightarrow B) \rightarrow (\Box A \supset \Box B)$ .  
Same method as 73 using 18.53 in place of 18.51.
75.  $\vdash (A \rightarrow B) \rightarrow (\sim \Diamond B \supset \sim \Diamond A)$ .  
Same method as 73 using 18.5.
76.  $\vdash (A \equiv B) \rightarrow (\Diamond A \equiv \Diamond B)$ .  
 $((A \rightarrow B) \rightarrow (\Diamond A \supset \Diamond B))((B \rightarrow A) \rightarrow (\Diamond B \supset \Diamond A))$  73, adj  
 $((A \rightarrow B)(B \rightarrow A)) \rightarrow ((\Diamond A \supset \Diamond B)(\Diamond B \supset \Diamond A))$  19.68, mod pon  
 $(A \equiv B) \rightarrow (\Diamond A \equiv \Diamond B)$  def
77.  $\vdash (A \equiv B) \rightarrow (\Box A \equiv \Box B)$ .  
Like 76, using 74.
78.  $\vdash (A \equiv B) \rightarrow (\sim \Diamond A \equiv \sim \Diamond B)$ .  
Like 76 using 75.
79.  $\vdash (A \equiv B) \equiv (\sim A \equiv \sim B)$ . 12.1, 12.44, subst, 12.15, def
80.  $\vdash ((A \equiv B)(\Gamma \equiv H)) \rightarrow ((A\Gamma) \equiv (BH))$ .  
19.68, adj, IX, 12.5, 12.15, subst
81.  $\vdash (A \equiv B) \rightarrow ((A\Gamma) \equiv (B\Gamma))$ . 19.6, adj, 19.68, mod pon, def
82.  $\vdash \Box (A \equiv B) \equiv (A \equiv B)$ .  
 $\Box(A \supset B) \equiv (A \rightarrow B)$  18.7  
 $\Box(B \supset A) \equiv (B \rightarrow A)$  18.7  
 $(\Box(A \supset B) \Box(B \supset A)) \rightarrow ((A \rightarrow B)(B \rightarrow A))$  IX  
 $\Box(A \equiv B) \equiv (A \equiv B)$  19.81, subst, def

Let every wff occurring in  $A$ , as well as  $A$  itself, be said to be within the scope of the left-hand diamond in the expression  $\Diamond A$ . An “ $n$ th degree occurrence of  $\Gamma$  in  $A$ ” will be an occurrence of  $\Gamma$  which falls within the scope of not more than  $n$  diamonds in  $A$ . Abbreviate  $A$  as  $\Box^0 A$  and  $\Box \Box^{n-1} A$  as  $\Box^n A$ .

XIX. If the wff's  $\Gamma, E, B$  are such that  $B$  results from  $A$  by the substitution of  $E$  for zero or more  $n$ th degree occurrences of  $\Gamma$  in  $A$  and if  $\alpha_1, \alpha_2, \dots, \alpha_m$  is a complete list of the free variables in  $\Gamma$  and  $E$ , then

$$\vdash \Box^n (\alpha_1)(\alpha_2) \dots (\alpha_m)(\Gamma \equiv E) \rightarrow (A \equiv B).$$

Proof by induction on  $n$ . When  $n$  is zero XIX becomes

$$\vdash (\alpha_1)(\alpha_2) \dots (\alpha_m)(\Gamma \equiv E) \rightarrow (A \equiv B), \text{ where as above,}$$

which we will call XIX<sub>0</sub>.

XIX<sub>0</sub> is provable by an induction on the number of occurrences of the primitive symbols  $\sim, \cdot, (\exists), \Diamond$  in  $A$ .

Suppose the number of occurrences of these symbols in  $A$  is zero. Then, XIX<sub>0</sub> is one of the following:

- (1).  $\vdash (\alpha_1)(\alpha_2) \dots (\alpha_m)(\Gamma \equiv E) \rightarrow (\Gamma \equiv E)$
- (2).  $\vdash (\alpha_1)(\alpha_2) \dots (\alpha_m)(\Gamma \equiv E) \rightarrow (A \equiv A)$ .

Case (1) is provable by successive applications of 8 and VIII.

$$\begin{aligned}
 \text{Case (2).} \quad & \vdash \sim \diamond (A \sim A) \rightarrow ((A \sim A) \rightarrow \sim (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)) && 19.74 \\
 & \vdash \sim \diamond (A \sim A) && 18.8 \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow \sim (A \sim A) \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (A \supset A) && \text{mod pon, X, 12.3, subst} \\
 & \vdash \square (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow \square (A \equiv A) && \text{def} \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (A \equiv A) && \text{VII, 12.7, def, subst} \\
 & && 72, 82, \text{subst}
 \end{aligned}$$

Assume that  $XIX_0$  is true when  $A$  contains not more than  $i$  occurrences of  $\sim$ ,  $(\exists)$ , and  $\diamond$ . If  $A$  contains  $i + 1$  of these symbols it must be  $\sim A_1$ ,  $A_1 \cdot A_2$ ,  $(\exists \alpha)A_1$  or  $\diamond A_1$ .  $XIX_0$  becomes correspondingly

$$\begin{aligned}
 (1.1). \quad & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (\sim A_1 \equiv \sim B_1), \\
 (2.1). \quad & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow ((A_1 \cdot A_2) \equiv (B_1 \cdot B_2)), \\
 (3.1). \quad & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow ((\exists \alpha)A_1 \equiv (\exists \alpha)B_1), \\
 (4.1). \quad & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (\diamond A_1 \equiv \diamond B_1),
 \end{aligned}$$

where  $B_1$  and  $B_2$  result from the substitution of  $E$  for zero or more zero degree occurrences of  $\Gamma$  in  $A_1$  and  $A_2$  respectively.

The following proof assumes that  $\Gamma$  is not identical with  $A$ . If it is, the proof is the same as Case (1) above. Where  $\Gamma$  does not occur in  $A$  the proof is like that of Case (2).

Case (1.1) follows immediately from the hypothesis, 79, and subst.

$$\begin{aligned}
 \text{Case (2.1).} \quad & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (A_1 \equiv B_1) && \text{hyp} \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (A_2 \equiv B_2) && \text{hyp} \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow ((A_1 \equiv B_1) (A_2 \equiv B_2)) && \text{XII} \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow ((A_1 \cdot A_2) \equiv (B_1 \cdot B_2)) && 80, \text{VIII}
 \end{aligned}$$

$$\begin{aligned}
 \text{Case (3.1).} \quad & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (A_1 \equiv B_1) && \text{hyp} \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (\alpha)(A_1 \equiv B_1) && \text{hyp, XIII} \\
 & (\alpha)(A_1 \equiv B_1) \rightarrow ((\exists \alpha)A_1 \equiv (\exists \alpha)B_1) && 44 \\
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow ((\exists \alpha)A_1 \equiv (\exists \alpha)B_1) && \text{VIII}
 \end{aligned}$$

Case (4.1). On the hypothesis  $n$  is zero, no occurrence of  $\Gamma$  in  $A$  is replaced by  $E$ . This case then becomes

$$\begin{aligned}
 & \vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (\diamond A_1 \equiv \diamond A_1) \\
 & \text{Proof like Case (2).}
 \end{aligned}$$

Suppose  $XIX$  is provable for  $n = k$ . We shall then prove it for  $n = k + 1$ . We will assume that  $\Gamma$  is not identical with  $A$  and that it occurs in  $A$ . If  $\Gamma$  is identical with  $A$  the proof is like Case 1) using in addition, successive applications of 18.42. If  $\Gamma$  does not occur in  $A$  the proof is like Case (2), using  $\square^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$  in place of  $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$ .

First it is necessary to prove the case where  $A$  is  $\diamond A_1$ ,  $B$  is  $\diamond B_1$ , and where

$A_1$  results from  $B_1$  by replacing by  $E$  zero or more  $k$ th degree occurrences of  $\Gamma$  in  $A$ . The other cases follow from an induction on the number of "additional occurrences" in  $A$ . An "additional occurrence" in  $A$  is an occurrence (of a primitive symbol) which is not part of a wff in  $A$  of the form  $\diamond H$ .

$$\begin{array}{l}
 \text{Case (1.2). } \vdash \Box^k(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (A_1 \equiv B_1) \quad \text{hyp} \\
 \vdash (A_1 \equiv B_1) \rightarrow (\diamond A_1 \equiv \diamond B_1) \quad 76 \\
 \vdash \Box^k(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (\diamond A_1 \equiv \diamond B_1) \quad \text{VIII} \\
 \vdash \Box \Box^k(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow \Box(\diamond A_1) \equiv \diamond B_1 \quad \text{VII} \\
 \vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (\diamond A_1 \equiv \diamond B_1) \\
 \text{def, 72, 82, subst}
 \end{array}$$

Case (1.2) is the proof of XIX for zero additional occurrences of the primitive symbols in  $A$ . Suppose XIX is true for not more than  $i$  additional occurrences of the primitive symbols in  $A$ . We will now show that it is true for  $i + 1$  additional occurrences.

If  $A$  contains  $i + 1$  additional occurrences it must be  $\sim A_1$ ,  $A_1 \cdot A_2$ , or  $(\exists \alpha)A_1$ . XIX becomes correspondingly

$$\begin{array}{l}
 (1.3). \quad \vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (\sim A_1 \equiv \sim B_1), \\
 (2.3). \quad \vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow ((A_1 \cdot A_2) \equiv (B_1 \cdot B_2)), \\
 (3.3). \quad \vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow ((\exists \alpha)A_1 \equiv (\exists \alpha)B_1),
 \end{array}$$

where  $B_1$  and  $B_2$  result from the substitution of  $E$  for zero or more  $k + 1$  degree occurrences of  $\Gamma$  in  $A_1$  and  $A_2$  respectively.

The proofs of (1.3), (2.3), and (3.3) are like those of Cases (1.1), (2.1), and (3.1) replacing  $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$  by  $\Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$ .

The addition of the following axiom schema,  $\diamond \diamond A \rightarrow \diamond A$ , would make it possible to obtain  $\Box \Box A \equiv \Box A$  and hence to prove XIX\* as follows:

XIX\*. If the wff's  $\Gamma$ ,  $E$ , and  $B$  are such that  $B$  results from  $A$  by the substitution of  $E$  for zero or more occurrences of  $\Gamma$  in  $A$  and if  $\alpha_1, \alpha_2, \dots, \alpha_m$  is a complete list of the free variables in and  $E$  then

$$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \rightarrow (A \equiv B).$$

The assumption  $\diamond \diamond A \rightarrow \diamond A$  is the distinctive feature of  $S4^8$  in contrast to  $S2$ , and if it were included in the present system we could prove XIX\* and thus dispense with a consideration of the "degree" of occurrence of  $\Gamma$  in  $A$ .

It might be interesting to note some of the differences between this system and the ordinary functional calculus of first order.

The converses of 47 and 48,

$$\begin{array}{l}
 ((\alpha)B \rightarrow A) \rightarrow (\exists \alpha)(B \rightarrow A), \text{ where } \alpha \text{ is not free in } A, \\
 ((A \rightarrow (\exists \alpha)B) \rightarrow (\exists \alpha)(A \rightarrow B), \text{ where } \alpha \text{ is not free in } A,
 \end{array}$$

<sup>8</sup> See Lewis and Langford, p. 501.



are not provable although their analogues hold in a functional calculus based on material implication.

The theorems corresponding to the so-called paradoxes of material implication such as

$$\sim(\exists\alpha)A \rightarrow (\alpha)(A \rightarrow B)$$

are not provable. However we can derive

$$\sim\Diamond(\exists\alpha)A \rightarrow (\alpha)(A \rightarrow B).$$

Theorems like

$$(\alpha)(A \rightarrow (B \vee \Gamma)) \rightarrow ((\alpha)(A \rightarrow B) \vee (\exists\alpha)(A\Gamma))$$

depend on some such principle as

$$(A \rightarrow (B \vee \Gamma)) \rightarrow (A \rightarrow B) \vee (A \rightarrow \Gamma),$$

which is not available in S2.

We cannot derive

$$(\alpha)(A \rightarrow (B \vee \Gamma)) \equiv (B \vee (\alpha)(A \rightarrow \Gamma)), \text{ where } \alpha \text{ is not free in } B,$$

although

$$(\alpha)(A \rightarrow (B \vee \Gamma)) \equiv \sim\Diamond\sim(B \vee (\alpha)(A \supset \Gamma)), \text{ where etc.}$$

is a theorem.<sup>9</sup>

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