On the Algebra of Logic [Continued]
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and as far as possible the $\Sigma$'s should be carried to the left of the $\Pi$'s. We have

$$\Pi_i \Pi_j x_{ij} = \Pi_j \Pi_i x_{ij}$$

$$\Sigma_i \Sigma_j x_{ij} = \Sigma_j \Sigma_i x_{ij}$$

and also

$$\Sigma_i \Pi_j x_i y_j = \Pi_j \Sigma_i x_i y_j .$$

But this formula does not hold when the $i$ and $j$ are not separated. We do have, however,

$$\Sigma_i \Pi_j x_{ij} \prec \Pi_i \Sigma_j x_{ij} .$$

It will, therefore, be well to begin by putting the $\Sigma$'s to the left, as far as possible, because at a later stage of the work they can be carried to the right but not to the left. For example, if the operators of the two premises are $\Pi_i \Sigma_j \Pi_k$ and $\Sigma_x \Pi_y \Sigma_z$, we can unite them in either of the two orders

$$\Sigma_x \Pi_y \Sigma_z \Pi_i \Sigma_j \Pi_k$$

$$\Sigma_x \Pi_i \Sigma_j \Pi_y \Sigma_z \Pi_k ,$$

and shall usually obtain different conclusions accordingly. There will often be room for skill in choosing the most suitable arrangement.

3d. It is next sometimes desirable to manipulate the Boolian part of the expression, and the letters to be eliminated can, if desired, be eliminated now. For this purpose they are replaced by relations of second intention, such as "other than," etc. If, for example, we find anywhere in the expression

$$a_{ijk} \bar{a}_{xyz} ,$$

this may evidently be replaceable by

$$(n_{ix} + n_{jy} + n_{kz})$$

where, as usual, $n$ means not or other than. This third step of the process is frequently quite indispensable, and embraces a variety of processes; but in ordinary cases it may be altogether dispensed with.

4th. The next step, which will also not commonly be needed, consists in making the indices refer to the same collections of objects, so far as this is useful. If the quantifying part, or Quantifier, contains $\Sigma_x$, and we wish to replace the $x$ by a new index $i$, not already in the Quantifier, and such that every $x$ is an $i$, we can do so at once by simply multiplying every letter of the Boolian having $x$ as an index by $x_i$. Thus, if we have "some woman is an angel" written in the form $\Sigma_w a_w$ we may replace this by $\Sigma_i (a_i w_i)$. It will be more often useful to replace the index of a $\Pi$ by a wider one; and this will be done by adding $\bar{x}_i$ to every letter having $x$ as an index. Thus, if we have "all dogs are animals, and all animals are vertebrates" written thus

$$\Pi_d a_d \ \Pi_a v_a ,$$

where $a$ and $\alpha$ alike mean animal, it will be found convenient to replace the last index by $i$, standing for any object, and to write the proposition

$$\Pi_i(\bar{\alpha}_i + v_i).$$

5th. The next step consists in multiplying the whole Boolian part, by the modification of itself produced by substituting for the index of any $\Pi$ any other index standing to the left of it in the Quantifier. Thus, for

$$\Sigma_i \Pi_j l_{ij},$$

we can write
$$\Sigma_i \Pi_j l_{ij} l_{ii}.$$

6th. The next step consists in the re-manipulation of the Boolian part, consisting, 1st, in adding to any part any term we like; 2d, in dropping from any part any factor we like, and 3d, in observing that

$$x\bar{x} = \mathbf{f}, \qquad x + \bar{x} = \mathbf{v},$$

so that
$$x\bar{x}y + z = z \quad (x + \bar{x} + y)\, z = z.$$

7th. $\Pi$'s and $\Sigma$'s in the Quantifier whose indices no longer appear in the Boolian are dropped.

The fifth step will, in practice, be combined with part of the sixth and seventh. Thus, from $\Sigma_i \Pi_j l_{ij}$ we shall at once proceed to $\Sigma_i l_{ii}$ if we like.

The following examples will be sufficient.

From the premises $\Sigma_i a_i b_i$ and $\Pi_j(\bar{b}_j + c_j)$, eliminate $b$. We first write

$$\Sigma_i \Pi_j a_i b_i (\bar{b}_j + c_j).$$

The distributive process gives

$$\Sigma_i \Pi_j a_i (b_i \bar{b}_j + b_i c_j).$$

But *we always have a right to drop a factor or insert an additive term.* We thus get
$$\Sigma_i \Pi_j a_i (b_i \bar{b}_j + c_j).$$

By the third process, we can, if we like, insert $n_{ij}$ for $b_i \bar{b}_j$. In either case, we identify $j$ with $i$ and get the conclusion

$$\Sigma_i a_i c_i.$$

Given the premises
$$\Sigma_h \Pi_i \Sigma_j \Pi_k (\alpha_{hik} + s_{jk} l_{ji})$$
$$\Sigma_u \Sigma_v \Pi_x \Pi_y (\varepsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Required to eliminate $s$. The combined premise is

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x \Pi_k \Pi_y (\alpha_{hik} + s_{jk} l_{ji})(\varepsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Identify $k$ with $v$ and $y$ with $j$, and we get

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x (\alpha_{hiv} + s_{jv} l_{ji})(\varepsilon_{ujx} + \bar{s}_{jv} b_{vx}).$$

The Boolian part then reduces, so that the conclusion is

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x (\alpha_{hiv} \varepsilon_{ujx} + \alpha_{hiv} b_{vx} + \varepsilon_{ujx} l_{ji}).$$

### IV.—*Second-intentional Logic.*

Let us now consider the logic of terms taken in collective senses. Our notation, so far as we have developed it, does not show us even how to express that two indices, $i$ and $j$, denote one and the same thing. We may adopt a special token of second intention, say 1, to express identity, and may write $1_{ij}$. But this relation of identity has peculiar properties. The first is that if $i$ and $j$ are identical, whatever is true of $i$ is true of $j$. This may be written

$$\Pi_i \Pi_j \{ \bar{1}_{ij} + \bar{x}_i + x_j \}.$$

The use of the general index of a token, $x$, here, shows that the formula is iconical. The other property is that if everything which is true of $i$ is true of $j$, then $i$ and $j$ are identical. This is most naturally written as follows: Let the token, $q$, signify the relation of a quality, character, fact, or predicate to its subject. Then the property we desire to express is

$$\Pi_i \Pi_j \Sigma_k (1_{ij} + \bar{q}_{ki} q_{kj}).$$

And identity is defined thus $1_{ij} = \Pi_k (q_{ki} q_{kj} + \bar{q}_{ki} \bar{q}_{kj})$.

That is, to say that things are identical is to say that every predicate is true of both or false of both. It may seem circuitous to introduce the idea of a quality to express identity; but that impression will be modified by reflecting that $q_{ki} q_{jk}$ merely means that $i$ and $j$ are both within the class or collection $k$. If we please, we can dispense with the token $q$, by using the index of a token and by referring to this in the Quantifier just as subjacent indices are referred to. That is to say, we may write $1_{ij} = \Pi_x (x_i x_j + \bar{x}_i \bar{x}_j)$.

The properties of the token $q$ must now be examined. These may all be summed up in this, that taking any individuals $i_1$, $i_2$, $i_3$, etc., and any individuals, $j_1, j_2, j_3$, etc., there is a collection, class, or predicate embracing all the $i$'s and excluding all the $j$'s except such as are identical with some one of the $i$'s. This might be written

$$(\Pi_\alpha \Pi_{i_\alpha})(\Pi_\beta \Pi_{j_\beta}) \Sigma_k (\Pi_\alpha \Sigma_{i'_\alpha}) \Pi_l \;\; q_{k i_\alpha} (\bar{q}_{k j_\beta} + q_{l i'_\alpha} q_{l j_\beta} + \bar{q}_{l i'_\alpha} \bar{q}_{l j_\beta}),$$

where the $i$'s and the $i'$'s are the same lot of objects. This notation presents indices of indices. The $\Pi_\alpha \Pi_{i_\alpha}$ shows that we are to take any collection whatever of $i$'s, and then any $i$ of that collection. We are then to do the same with the $j$'s. We can then find a quality $k$ such that the $i$ taken has it, and also such that the $j$ taken wants it unless we can find an $i$ that is identical with the $j$ taken. The necessity of some kind of notation of this description in treating of classes collectively appears from this consideration: that in such discourse we are neither speaking of a single individual (as in the non-relative logic) nor of a small number of individuals considered each for itself, but of a whole class, perhaps an infinity of individuals. This suggests a relative term with an indefinite series of indices as $x_{ijkl\ldots}$. Such a relative will, however, in most, if not in all cases, be of a degenerate kind and is consequently expressible as above. But it seems preferable to attempt a partial decomposition of this definition. In the first place, any individual may be considered as a class. This is written

$$\Pi_i \Sigma_k \Pi_j \ q_{ki}(\bar{q}_{kj} + 1_{ij}).$$

This is the *ninth icon.* Next, given any class, there is another which includes all the former excludes and excludes all the former includes. That is,

$$\Pi_l \Sigma_k \Pi_i (q_{li}\bar{q}_{ki} + \bar{q}_{li}q_{ki}).$$

This is the *tenth icon.* Next, given any two classes, there is a third which includes all that either includes and excludes all that both exclude. That is

$$\Pi_l \Pi_m \Sigma_k \Pi_i (q_{li}q_{ki} + q_{mi}q_{ki} + \bar{q}_{li}\bar{q}_{mi}\bar{q}_{ki}).$$

This is the *eleventh icon.* Next, given any two classes, there is a class which includes the whole of the first and any one individual of the second which there may be not included in the first and nothing else. That is,

$$\Pi_l \Pi_m \Pi_i \Sigma_k \Pi_j \{ q_{li} + \bar{q}_{mi} + q_{ki}(q_{kj} + \bar{q}_{lj}) \}.$$

This is the *twelfth icon.*

To show the manner in which these formulæ are applied let us suppose we have given that everything is either true of $i$ or false of $j$. We write

$$\Pi_k (q_{ki} + \bar{q}_{kj}).$$

The tenth icon gives $\quad \Pi_l \Sigma_k (q_{li}\bar{q}_{ki} + \bar{q}_{li}q_{ki})(q_{lj}\bar{q}_{kj} + \bar{q}_{lj}q_{kj})$

Multiplication of these two formulæ give

$$\Pi_l \Sigma_k (q_{ki}\bar{q}_{li} + q_{lj}\bar{q}_{kj}),$$

or, dropping the terms in $k$ $\quad \Pi_l (\bar{q}_{li} + q_{lj})$.

Multiplying this with the original datum and identifying $l$ with $k$, we have

$$\Pi_k (q_{ki}q_{kj} + \bar{q}_{ki}\bar{q}_{kj}).$$

No doubt, a much more direct method of procedure could be found.

Just as $q$ signifies the relation of predicate to subject, so we need another token, which may be written $r$, to signify the conjoint relation of a simple relation, its relate and its correlate. That is, $r_{j\alpha i}$ is to mean that $i$ is in the relation $\alpha$ to $j$. Of course, there will be a series of properties of $r$ similar to those of $q$. But it is singular that the uses of the two tokens are quite different. Namely, the chief use of $r$ is to enable us to express that the number of one class is at least as great as that of another. This may be done in a variety of different ways. Thus, we may write that for every $a$ there is a $b$, in the first place, thus:

$$\Sigma_\alpha \Pi_i \Sigma_j \Pi_h \{\bar{a}_i + b_j r_{j\alpha i}(\bar{r}_{j\alpha h} + \bar{a}_h + 1_{ih})\}.$$

But, by an icon analogous to the eleventh, we have

$$\Pi_\alpha \Pi_\beta \Sigma_\gamma \Pi_u \Pi_v (r_{u\alpha v} r_{u\gamma v} + r_{u\beta v} r_{u\gamma v} + \bar{r}_{u\alpha v} \bar{r}_{u\beta v} \bar{r}_{u\gamma v}).$$

From this, by means of an icon analogous to the *tenth*, we get the general formula

$$\Pi_\alpha \Pi_\beta \Sigma_\gamma \Pi_u \Pi_v \{r_{u\alpha v} r_{u\beta v} r_{u\gamma v} + \bar{r}_{u\gamma v}(\bar{r}_{u\alpha v} + \bar{r}_{u\beta v})\}.$$

For $r_{u\beta v}$ substitute $a_u$ and multiply by the formula the last but two. Then, identifying $u$ with $h$ and $v$ with $j$, we have

$$\Sigma_\alpha \Pi_i \Sigma_h \Pi_h \{\bar{a}_i + b_j r_{j\alpha i}(\bar{r}_{j\alpha h} + 1_{ih})\}$$

a somewhat simpler expression. However, the best way to express such a proposition is to make use of the letter $c$ as a token of a one-to-one correspondence. That is to say, $c$ will be defined by the three formulæ,

$$\Pi_\alpha \Pi_u \Pi_v \Pi_w (\bar{c}_\alpha + \bar{r}_{u\alpha v} + \bar{r}_{u\alpha w} + 1_{vw})$$
$$\Pi_\alpha \Pi_u \Pi_v \Pi_w (\bar{c}_\alpha + \bar{r}_{u\alpha w} + r_{v\alpha w} + 1_{uv})$$
$$\Pi_\alpha \Sigma_u \Sigma_v \Sigma_w (c_\alpha + r_{u\alpha v} r_{u\alpha w} \bar{1}_{vw} + r_{u\alpha w} r_{v\alpha w} \bar{1}_{uv}).$$

Making use of this token, we may write the proposition we have been considering in the form

$$\Sigma_\alpha \Pi_i \Sigma_j \; c_\alpha(\bar{a}_i + b_j r_{j\alpha i}).$$

In an appendix to his memoir on the logic of relatives, DeMorgan enriched the science of logic with a new kind of inference, the syllogism of transposed quantity. DeMorgan was one of the best logicians that ever lived and unquestionably the father of the logic of relatives. Owing, however, to the imperfection of his theory of relatives, the new form, as he enunciated it, was a down-right paralogism, one of the premises being omitted. But this being supplied, the form furnishes a good test of the efficacy of a logical notation. The following is one of DeMorgan's examples:

Some $X$ is $Y$,
For every $X$ there is something neither $Y$ nor $Z$;
Hence, something is neither $X$ nor $Z$.

The first premise is simply $\Sigma_a x_a y_a$.
The second may be written

$$\Sigma_\alpha \Pi_i \Sigma_j \; c_\alpha(\bar{x}_i + r_{j\alpha i} \bar{y}_j \bar{z}_j).$$

From these two premises, little can be inferred. To get the above conclusion it is necessary to add that the class of $X$'s is a finite collection; were this not necessary the following reasoning would hold good (the limited universe consisting of numbers); for it precisely conforms to DeMorgan's scheme.

> Some odd number is prime;
> Every odd number has its square, which is neither prime nor even;
> Hence, some number is neither odd nor even.*

Now, to say that a lot of objects is finite, is the same as to say that if we pass through the class from one to another we shall necessarily come round to one of those individuals already passed; that is, if every one of the lot is in any one-to-one relation to one of the lot, then to every one of the lot some one is in this same relation. This is written thus:

$$\Pi_\beta \Pi_u \Sigma_v \Sigma_s \Pi_t \{\bar{c}_\beta + \bar{x}_u + x_v r_{u\beta v} + x_s(\bar{x}_t + \bar{r}_{t\beta s})\}$$

Uniting this with the two premises and the second clause of the definition of $c$, we have

$$\Sigma_a \Sigma_\alpha \Pi_\beta \Pi_u \Sigma_v \Sigma_s \Pi_i \Sigma_j \Pi_t \Pi_\gamma \Pi_e \Pi_f \Pi_g \, x_a y_a c_\alpha (\bar{x}_i + r_{j\alpha i} \bar{y}_j \bar{z}_j)$$
$$\{\bar{c}_\beta + \bar{x}_u + x_v r_{u\beta v} + x_s(\bar{x}_t + \bar{r}_{t\beta s})\}(\bar{c}_\gamma + \bar{r}_{e\gamma g} + \bar{r}_{f\gamma w} + 1_{ef}).$$

We now substitute $\alpha$ for $\beta$ and for $\gamma$, $a$ for $u$ and for $e$, $j$ for $t$ and for $f$, $v$ for $g$. The factor in $i$ is to be repeated, putting first $s$ and then $v$ for $i$. The Boolian part thus reduces to

$$(\bar{x}_s + r_{j\alpha s} \bar{y}_j \bar{z}_j) \, c_\alpha x_a y_a r_{a\alpha v} x_v r_{j\alpha v} \bar{y}_j \bar{z}_j 1_{aj} + r_{j\alpha s} \bar{y}_j \bar{z}_j x_s \bar{x}_j (\bar{x}_v + r_{j\alpha v} \bar{y}_j \bar{z}_j)(\bar{r}_{a\alpha v} + \bar{r}_{j\alpha v} + 1_{aj}),$$

which, by the omission of factors, becomes

$$y_a \bar{y}_j 1_{aj} + \bar{x}_j \bar{z}_j.$$

Thus we have the conclusion $\qquad \Sigma_j \bar{x}_j \bar{z}_j.$

It is plain that by a more iconical and less logically analytical notation this procedure might be much abridged. How minutely analytical the present system is, appears when we reflect that every substitution of indices of which nine were used in obtaining the last conclusion is a distinct act of inference. The annulling of $(y_a \bar{y}_j 1_{aj})$ makes ten inferential steps between the premises and conclusion of the syllogism of transposed quantity.

* Another of DeMorgan's examples is this: "Suppose a person, on reviewing his purchases for the day, finds, by his counterchecks, that he has certainly drawn as many checks on his banker (and maybe more) as he has made purchases. But he knows that he paid some of his purchases in money, or otherwise than by checks. He infers then that he has drawn checks for something else except that day's purchases. He infers rightly enough." Suppose, however, that what happened was this: He bought something and drew a check for it; but instead of paying with the check, he paid cash. He then made another purchase for the same amount, and drew another check. Instead, however, of paying with that check, he paid with the one previously drawn. And thus he continued without cessation, or *ad infinitum.* Plainly the premises remain true, yet the conclusion is false.